CONDITIONAL PROBABILITY

Lecture Notes

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Conditional Probability

For the general definition, take events A and B, and assume that $P(B) > 0$. The conditional probability of A given B equals

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Toss two fair coins, blindfolded. Somebody tells you that you tossed at least one *Heads*. What is the probability that both tosses are *Heads*?

- Here $A = \{both\ H\}$, $B = \{at\ least\ one\ H\}$,
- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{3}{4}} = \frac{1}{3}$

Toss a coin 10 times. If you know *(a)* that exactly 7 *Heads* are tossed, *(b)* that at least 7 *Heads* are tossed, what is the probability that your first toss is *Heads*?

(a) $A = \{\textit{First toss is Heads}\}$, $B = \{\textit{Exactly 7 Heads are tossed}\}$

- $$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{2}\binom{9}{6}\left(\frac{1}{2}\right)^{6+3}}{\binom{10}{7}\left(\frac{1}{2}\right)^{7+3}} = \frac{7}{10}$$

(b) $A = \{\textit{First toss is Heads}\}$, $B = \{\textit{At least 7 Heads are tossed}\}$

$$P(B) = \left[\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right]\left(\frac{1}{2}\right)^{10}$$

- $$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\binom{9}{6}}{\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10}} = \frac{65}{88}$$

An urn contains 10 *black* and 10 *white* balls. Draw 3 *(a)* without replacement, and *(b)* with replacement. What is the probability that all three are *white*?

(a) $$P(3 \text{ are white}) = \frac{\binom{10}{3}}{\binom{20}{3}}$$

$A_1 = 1^{st}$ *is white*, $A_2 = 2^{nd}$ *is white*, $A_3 = 3^{rd}$ *is white*.
$P(A_1) = \frac{1}{2}$, $P(A_2|A_1) = \frac{9}{19}$, $P(A_3|A_2 \cap A1) = \frac{8}{18}$
$P(A_3|A_2 \cap A_1) = \frac{P(A_3 \cap A_2 \cap A_1)}{P(A_2 \cap A_1)} = \frac{P(A_3 \cap A_2 \cap A_1)}{P(A_2|A_1)P(A_1)}$

- $P(A_3 \cap A_2 \cap A_1) = P(A_3|A_2 \cap A_1)P(A_2|A_1)P(A_1)$
$= \frac{8}{18} \cdot \frac{9}{19} \cdot \frac{1}{2}$

(b) $P(A_3 \cap A_2 \cap A_1) = \left(\frac{1}{2}\right)^3$

Theorem 4.1. First Bayes' formula

Assume that $F_1, \ldots, F_n$ are pairwise disjoint and that $F_1 \cup \cdots \cup F_n = \Omega$, that is, exactly one of them always happens. Then, for an event A,
$P(A) = P(F1)P(A|F_1) + P(F_2)P(A|F_2) + \cdots + P(F_n)P(A|F_n)$.

Flip a fair coin. If you toss *Heads*, roll 1 die. If you toss *Tails*, roll 2 dice. Compute the probability that you roll exactly one 6.

A: {Roll exactly one 6}. F_1: {Toss *Heads*}, F_2: {Toss *Tails* }.

- $P(A) = P(F1)P(A|F_1) + P(F_2)P(A|F_2)$
  $P(A) = \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot 2 = \frac{2}{9}$

Roll a die, then select at random, without replacement, as many cards from deck as the number shown on the die. What is the probability that you get at least one *Ace*?

F_i: $\{$number shown on the die is $i\}$, for $i = 1, \cdots, 6$.
A: $\{$ no Ace is chosen $\}$.

$$P(A) = \sum_{i=1}^{6} P(F_i)P(A|F_i)$$

$$P(F_i) = \frac{1}{6}$$

$$P(A|F_1) = \frac{48}{52},\ P(A|F_2) = \frac{48}{52} \cdot \frac{47}{51},\ \ldots,\ P(A|F_6) = \prod_{i=0}^{5} \frac{48-i}{52-i}$$

- $P(A^c) = 1 - P(A) = \left(1 - \frac{1}{6} \sum_{i=1}^{6} \frac{\binom{48}{i}}{\binom{52}{i}}\right)$

Theorem 4.2. Second Bayes' formula

Let $F_1, \ldots, F_n$ and A be as in Theorem 4.1. Then

$$P(F_j|A) = \frac{P(F_j \cap A)}{P(A)} = \frac{P(A|F_j)P(F_j)}{P(A|F_1)P(F_1) + \cdots + P(A|F_n)P(F_n)}$$

An event F_j is often called a *hypothesis*, $P(F_j)$ its prior probability, and $P(F_j|A)$ its posterior probability.

We have a fair coin and an unfair coin, which always comes out *Heads*. Choose one at random, toss it twice. It comes out *Heads* both times. What is the probability that the coin is fair?

$$P(Fair|Heads\ both) = \frac{P(Heads\ both|Fair)P(Fair)}{P(Heads\ both|Fair)P(Fair)+P(Heads\ both|Unfair)P(Unfair)}$$

- $P(Fair|Heads\ both) = \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+1\cdot1\cdot\frac{1}{2}} = \frac{1}{5}$

A factory has three machines, M_1, M_2 and M_3, that produce items (say, lightbulbs). It is impossible to tell which machine produced a particular item, but some are defective. Here are the known numbers: You pick an item, test it, and find it is defective. What is the probability that it was made by machine M_2?

machine	proportion of items made	prob. any made item is defective
M_1	0.2	0.001
M_2	0.3	0.002
M_3	0.5	0.003

$$P(M_2|Defective) = \frac{P(Defective|M_2)P(M_2)}{P(Defective|M_1)P(M_1)+P(Defective|M_2)P(M_2)+P(Defective|M_3)P(M_3)}$$

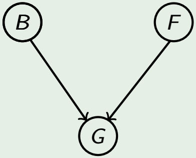
- $P(M_2|Defective) = \frac{2\cdot 10^{-3}\cdot 0.3}{1\cdot 10^{-3}\cdot 0.2+2\cdot 10^{-3}\cdot 0.3+3\cdot 10^{-3}\cdot 0.5} \approx 0.26$

Assume 10% of people have a certain disease. A test gives the correct diagnosis with probability of 0.8; that is, if the person is sick, the test will be positive with probability 0.8, but if the person is not sick, the test will be positive with probability 0.2. A random person from the population has tested positive for the disease. What is the probability that he is actually sick?

$$P(Sick|Positive) = \frac{P(Positive|Sick)P(Sick)}{P(Positive|Sick)P(Sick)+P(Positive|Not\ Sick)P(Non\ Sick)}$$

- $P(Sick|Positive) = \frac{0.8\cdot 0.1}{0.8\cdot 0.1+0.2\cdot 0.9} = \frac{8}{26} \approx 0.31$ (**Not** 0.8**!**)

Suppose we observe the fuel gauge and discover that it reads empty *i.e.* $G = 0$. (a) What is the probability that the fuel tank is empty? (b) Additionally, if the battery also reads 0, what is the probability that the fuel tank is empty?



B : State of the Battery $\quad P(B = 1) = 0.9$

F : State of the Fuel Tank $\quad P(F = 1) = 0.9$

G : Reading on the Fuel Gauge

$P(G = 1|B = 1, F = 1) = 0.8$

$P(G = 1|B = 1, F = 0) = 0.2$

$P(G = 1|B = 0, F = 1) = 0.2$

$P(G = 1|B = 0, F = 0) = 0.1$

(a) $P(F = 0|G = 0) = \frac{P(F=0,G=0)}{p(G=0)} = \frac{P(G=0|F=0)P(F=0)}{p(G=0)}$

$$P(G = 0|F = 0) = \sum_B P(G = 0|F = 0, B)P(B) = 0.9 \cdot 0.1 + 0.8 \cdot 0.9 = 0.81$$

$$P(G = 0) = \sum_{B,F} P(G = 0|B, F)P(B, F) = \sum_{B,F} P(G = 0|B, F)P(B)P(F)$$

$$P(G = 0) = 0.9 \cdot 0.1 \cdot 0.1 + 0.8 \cdot 0.1 \cdot 0.9 + 0.8 \cdot 0.9 \cdot 0.1 + 0.2 \cdot 0.9 \cdot 0.9 = 0.315$$

- $P(F = 0|G = 0) = \frac{0.81 \cdot 0.1}{0.315} \approx 0.257$

(b) $P(F = 0|G = 0, B = 0) = \frac{P(G=0|F=0,B=0)P(F=0)}{p(G=0|B=0)} = \frac{P(G=0|F=0,B=0)P(F=0)}{p(G=0|B=0)}$

$$P(G = 0|B = 0) = \sum_F P(G = 0|B = 0, F)P(F) = 0.9 \cdot 0.1 + 0.8 \cdot 0.9 = 0.81$$

- $P(F = 0|G = 0, B = 0) = \frac{0.9 \cdot 0.1}{0.81} \approx 0.111$

Events A and B are independent if $P(A \cap B) = P(A)P(B)$ and dependent (or correlated) otherwise.

Assuming that $P(B) > 0$, one could rewrite the condition for independence,

$$P(A|B) = P(A),$$

so the probability of A is unaffected by knowledge that B occurred. Also, if A and B are independent, $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$, so A and Bc are also independent knowing that B has not occurred also has no influence on the probability of A.

Pick a random card from a full deck. Let $A = \{card\ is\ an\ Ace\}$ and $R = \{card\ is\ red\}$. Are A and R independent?

- $P(A|R) = \frac{P(A \cap R)}{P(R)} = \frac{\frac{2}{52}}{\frac{26}{52}} = \frac{1}{13} = P(A) \longrightarrow A$ and R are independent.

Now, pick two cards out of the deck sequentially without replacement. Are $F = \{\textit{first card is an Ace}\}$ and $S = \{\textit{second card is an Ace}\}$ independent?

$P(S|F) = \frac{P(S \cap F)}{P(F)} = \frac{\frac{3}{51} \cdot \frac{4}{52}}{\frac{4}{52}} = \frac{3}{51} \neq P(S) = \frac{4}{52} = \frac{1}{13} \longrightarrow S$ and F are *not* independent.

Roll a four sided fair die, that is, choose one of the numbers $1, 2, 3, 4$ at random. Let $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$. Check that these are pairwise independent (each pair is independent), but not independent

$P(A \cap B) = P(\{1\}) = \frac{1}{4} = P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(B \cap C) = P(\{1\}) = \frac{1}{4} = P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(A \cap C) = P(\{1\}) = \frac{1}{4} = P(A)P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ They are pairwise independent.

$P(A \cap B \cap C) = P(\{1\}) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

You roll a die, your friend tosses a coin.

- If you roll 6, you win outright.
- If you do not roll 6 and your friend tosses *Heads*, you lose outright.
- If neither, the game is repeated until decided.

What is the probability that you win?

$$P(win) = P(win\ at\ the\ 1^{st}\ run) + P(win\ at\ the\ 2^{nd}\ run) + \dots$$
$$P(win) = \tfrac{1}{6} + \tfrac{5}{6} \cdot \tfrac{1}{2} \cdot \tfrac{1}{6} + (\tfrac{5}{6})^2 \cdot (\tfrac{1}{2})^2 \cdot \tfrac{1}{6} + \dots = \tfrac{1}{6} \sum_{n=0}^{\infty} (\tfrac{5}{12})^n = \tfrac{1}{6} \cdot \frac{1}{1-\frac{5}{12}} = \tfrac{2}{7} = 0.35$$

Important note:

We have implicitly assumed independence between the coin and the die, as well as between different tosses and rolls. This is very in problems such as this!

$D = \{game\ is\ decided\ on\ 1^{st}\ round\}$, $W = \{you\ win\}$.
Since D and W are independent, $P(W|D^c) = P(W)$ and $P(W) = P(W|D)$.
$P(W|D) = \frac{P(W \cap D)}{P(D)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{5}{6} \cdot \frac{1}{2}} = \frac{2}{7}$

Craps. Many casinos allow you to bet even money on the following game. Two dice are rolled and the sum S is observed.

- If $S \in \{7, 11\}$, you win immediately.
- If $S \in \{2, 3, 12\}$, you lose immediately.
- If $S \in \{4, 5, 6, 8, 9, 10\}$, the pair of dice is rolled repeatedly until one of the following happens:

  S repeats, in which case you win.

  7 appears, in which case you lose.

What is the winning probability?

$P(\textit{win on the } 1^{st} \textit{ roll}) = \frac{8}{36}$, $P(\textit{loss on the } 1^{st} \textit{ roll}) = \frac{4}{36}$, $P(\textit{loss after the } 1^{st} \textit{ roll}) = \frac{6}{36}$,

$P(4) = \frac{3}{36}$, $P(\textit{win with } 4) = \frac{P(4)}{P(4)+P(7)} = \frac{\frac{3}{36}}{\frac{3}{36}+\frac{6}{36}} = \frac{1}{3}$

$P(\textit{win with } 5) = \frac{\frac{4}{36}}{\frac{4}{36}+\frac{6}{36}} = \frac{2}{5}$

$P(\textit{win with } 6) = \frac{\frac{5}{36}}{\frac{5}{36}+\frac{6}{36}} = \frac{5}{11}$

$P(\textit{win with } 8) = \frac{\frac{5}{36}}{\frac{5}{36}+\frac{6}{36}} = \frac{5}{11}$

$P(\textit{win with } 9) = \frac{\frac{4}{36}}{\frac{4}{36}+\frac{6}{36}} = \frac{2}{5}$

$P(\textit{win with } 10) = \frac{\frac{3}{36}}{\frac{4}{36}+\frac{6}{36}} = \frac{2}{5}$

Using 1^{st} Bayes Formula

$P(win) = P(win|1^{st})P(1^{st}) + P(win|\textit{with } 4)P(4) + \ldots P(win|\textit{with } 10)P(10)$

$P(win) = \frac{8}{36} + 2 \cdot \left(\frac{1}{3} \cdot \frac{3}{36} + \frac{2}{5} \cdot \frac{4}{36} + \frac{5}{11} \cdot \frac{5}{36}\right) \approx 0.4929$

Bernoulli trials

Assume n independent experiments, each of which is a success with probability p and, thus, failure with probability $1 - p$.

In a sequence of n Bernoulli trials, $P(\textit{exactly } k \textit{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}$.

A machine produces items which are independently defective with probability p. Let us compute a few probabilities:

1. Probability that exactly two items among the first 6 are defective.

   $P = \binom{6}{2} p^2 (1-p)^4$

2. Probability that at least one item among the first 6 is defective.

   $P = 1 - \binom{6}{0} p^0 (1-p)^6 = 1 - (1-p)^6$

3. Probability that at least 2 items among the first 6 are defective.

   $P = 1 - \binom{6}{0} p^0 (1-p)^6 - \binom{6}{1} p^1 (1-p)^5$

4. Probability that exactly 100 items are made before 6 defective are found.

   $P = p \cdot \binom{99}{5} p^5 (1-p)^{94}$

Problem of Points. This involves finding the probability of n successes before m failures in a sequence of Bernoulli trials. Let us call this probability $p_{n,m}$.

$p_{n,m} = P(\textit{in the first } m+n-1 \textit{ trials}, \textit{ the number of successes is} \geq n)$

$$= \sum_{k=n}^{n+m-1} \binom{\text{n+m-1}}{\text{k}} p^k (1-p)^{n+m-1-k}$$

Assuming $m, n \geq 1$,
$p_{n,m} = P(\text{first trial is a success}) \cdot P(n-1 \text{ successes before } m \text{ failures})$
$+P(\text{first trial is a failure}) \cdot P(n \text{ successes before } m-1 \text{ failures})$

$$p_{n,m} = p \cdot p_{n-1,m} + (1-p) \cdot p_{n,n-1}$$

with $p_{n,0} = 0, \ p_{0,m} = 1$
which allows for very speedy and precise computations for large m and n.

Best of 7. Assume that two equally matched teams, A and B, play a series of games and that the first team that wins four games is the overall winner of the series. As it happens, team A lost the first game. What is the probability it will win the series? Assume that the games are Bernoulli trials with success probability $\frac{1}{2}$.

$m = 3$ and $n = 4$.

$$P = \sum_{k=4}^{6} \left(\begin{array}{c} 6 \\ \text{k} \end{array} \right) p^k (1-p)^{6-k} = \sum_{n=4}^{6} \left(\begin{array}{c} 6 \\ \text{k} \end{array} \right) (\tfrac{1}{2})^6 = \tfrac{15+6+1}{2^6} \approx 0.3438$$

Banach Matchbox Problem. A mathematician carries two matchboxes, each originally containing n matches. Each time he needs a match, he is equally likely to take it from either box. What is the probability that, upon reaching for a box and finding it empty, there are exactly k matches still in the other box? Here, $0 \leq k \leq n$.

After $n + n - k$ accesses, he has reached for box 1 exactly n times and he found it empty at the $(n + n - k) + 1^{st}$ trial.

$$P = 2 \cdot \tfrac{1}{2} \left(\begin{array}{c} \text{2n-k} \\ \text{n} \end{array} \right) p^n (1-p)^{2n-k-n} = \left(\begin{array}{c} \text{2n-k} \\ \text{n} \end{array} \right) (\tfrac{1}{2})^{2n-k}$$

Each day, you independently decide, with probability p, to flip a fair coin. Otherwise, you do nothing. (a) What is the probability of getting exactly 10 *Heads* in the first 20 days? (b) What is the probability of getting 10 *Heads* before 5 Tails?

(a) the probability of getting *Heads* is $\frac{p}{2}$ independently each day, so
$P = \begin{pmatrix} 20 \\ 10 \end{pmatrix} \left(\frac{p}{2}\right)^{10} \cdot \left(1 - \frac{p}{2}\right)^{10}$

(b) we can disregard days at which we do not flip to get *i.e.* $n = 10$ successes before $m = 5$ failures
$P = \sum_{k=10}^{14} \begin{pmatrix} 14 \\ \text{k} \end{pmatrix} \frac{1}{2^{14}}.$

Roll a die and your score is the number on the die. Your friend rolls 5 dice and his score is the number of 6's shown. Compute (a) the probability that the two scores are equal and (b) the probability that your friend's score is strictly larger than yours.

(a) $A = \{1, 2, \ldots, 6\}$ and $B = \{\text{number of 6's when rolling 5 dice }\}$.
$P(A = B) = P(B = A|A = 1)P(A = 1) + \cdots + P(B = A|A = 5)P(A = 5)$
$= \left[\begin{pmatrix} 5 \\ 1 \end{pmatrix} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^4\right] \cdot \frac{1}{6} + \cdots + \left[\begin{pmatrix} 5 \\ 5 \end{pmatrix} \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^0\right] \cdot \frac{1}{6} \approx 0.0997$

(b) $P(A < B) = P(A = 1|B > 1) \cdot P(B > 1) + \cdots + P(A = 4|B > 4) \cdot P(B > 4)$
$= \frac{1}{6}\left[\sum_{k=2}^{5} \begin{pmatrix} 5 \\ \text{k} \end{pmatrix} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{5-k} + \sum_{k=3}^{5} \begin{pmatrix} 5 \\ \text{k} \end{pmatrix} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{5-k} + \cdots + \begin{pmatrix} 5 \\ 5 \end{pmatrix} \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^0\right] \approx 0.0392.$

Graphs and Distributions

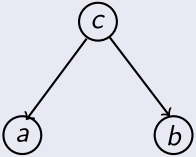
Conditional Independence

If $p(a|b,c) = p(a|c)$ then
a is *conditionally independent* of b given c shown as

$$a \perp\!\!\!\perp b|c$$

If $p(a,b|c) = p(a|b,c)p(b|c) = p(a|c)p(b|c)$
a and b are *statistically independent* given c.

tail-to-tail

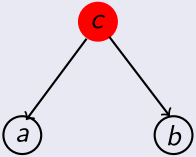


$$p(a,b,c) = p(a|c)p(b|c)p(c)$$

$$p(a,b) = \sum_c p(a|c)p(b|c)p(c)$$

$$a \not\perp\!\!\!\perp b | \varnothing$$

If none of the variables are observed, then we can investigate whether a and b are independent by marginalizing with respect to c.



$$p(a,b|c) = \frac{p(a,b,c)}{p(c)}$$

$$= p(a|c)p(b|c)$$

$$a \perp\!\!\!\perp b | c$$

When we condition on node c, the conditioned node 'blocks' the path from a to b and causes a and b to become (conditionally) independent.

head-to-tail

$$a \longrightarrow c \longrightarrow b$$

$$p(a,b,c) = p(a)p(c|a)p(b|c)$$

$$p(a,b) = p(a)\sum_c p(c|a)p(b|c)$$

$$= p(a)p(b|a)$$

$$a \not\perp\!\!\!\perp b|\emptyset$$

The node c is said to be head-to-tail with respect to the path from node a to node b. Such a path connects nodes a and b and renders them dependent.

$$a \longrightarrow c \longrightarrow b$$

$$p(a,b|c) = \frac{p(a,b,c)}{p(c)}$$

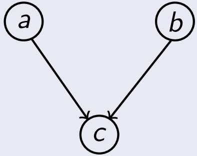
$$= \frac{p(a)p(c|a)p(b|c)}{p(c)}$$

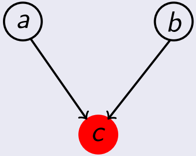
$$= p(a|c)p(b|c)$$

$$a \perp\!\!\!\perp b|c$$

If we now observe c, then this observation 'blocks' the path from a to b and so we obtain the conditional independence property.

head-to-head





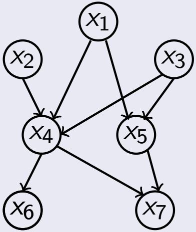
$$p(a,b,c) = p(a)p(b)p(c|a,b)$$
$$p(a,b) = p(a)p(b)$$
$$a \perp\!\!\!\perp b|\varnothing$$

$$p(a,b|c) = \frac{p(a,b,c)}{p(c)} = \frac{p(a)p(b)p(c|a,b)}{p(c)}$$
$$a \not\perp\!\!\!\perp b|c$$

As none of the variables are observed, by marginalizing over c we obtain a and b to be independent with no variables observed, in contrast to the two previous examples.

When we observe and condition on c, it 'unblocks' the path and renders a and b dependent.

Joint Probability Density using Directed Acyclical Graphs



$$p(x_1, x_2, x_3, x_4, x_5, x_6, x_7) =$$
$$p(x_1)p(x_2)p(x_3)p(x_4|x_1, x_2, x_3)p(x_5|x_1, x_3)$$
$$\cdot p(x_6|x_4)p(x_7|x_4, x_5)$$

Given the distribution in the table, Show if (a) $p(a,b) = p(a)p(b)$ and (b) $p(a,b|c) = p(a|c)p(b|c)$.

a	b	c	$p(a,b,c)$
0	0	0	0.192
0	0	1	0.144
0	1	0	0.048
0	1	1	0.216
1	0	0	0.192
1	0	1	0.064
1	1	0	0.048
1	1	1	0.096

(a) $p(a,b) = \sum_c p(a,b,c)$, $\quad p(a) = \sum_b p(a,b)$, $\quad p(b) = \sum_a p(a,b)$

a	b	$p(a,b)$	$p(a)$	$p(b)$	$p(a)p(b)$
0	0	0.3360	0.6000	0.5920	0.3552
0	1	0.2640	0.6000	0.4080	0.2448
1	0	0.2560	0.4000	0.5920	0.2368
1	1	0.1440	0.4000	0.4080	0.1632

$$p(a,b) \neq p(a)p(b)$$

(b) $p(a,b|c) = \frac{p(a,b,c)}{p(c)}$ $\quad p(c) = \sum_{a,b} p(a,b,c)$ $\quad p(a|c) = \sum_b p(a,b|c)$, $\quad p(b|c) = \sum_a p(a,b|c)$

a	b	c	$p(c)$	$p(a,b\|c)$	$p(a\|c)$	$p(b\|c)$	$p(a\|c)p(b\|c)$
0	0	0	0.4800	0.4000	0.5000	0.8000	0.4000
0	0	1	0.5200	0.2769	0.6923	0.4000	0.2769
0	1	0	0.4800	0.1000	0.5000	0.2000	0.1000
0	1	1	0.5200	0.4154	0.6923	0.6000	0.4154
1	0	0	0.4800	0.4000	0.5000	0.8000	0.4000
1	0	1	0.5200	0.1231	0.3077	0.4000	0.1231
1	1	0	0.4800	0.1000	0.5000	0.2000	0.1000
1	1	1	0.5200	0.1846	0.3077	0.6000	0.1846

$$p(a,b|c) = p(a|c)p(b|c)$$