$z-$Transform
Lecture Notes

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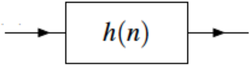
Some concepts and illustrations in this lecture are adapted from the textbook,

Signals and Systems, 2nd Edition by Alan Oppenheim, Alan Willisky and H. Nawab, *Prentice Hall*.

Eigenfunctions for Discrete LTI Systems

Eigenfunction : $z = re^{j\omega}$



z^n $\qquad$ $z^n H(z)$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n H(z)$$

$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$ is the $z-$transform of $h[n]$.

Example

$$a^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

$$-a^n u[-n-1] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \qquad |z| < |a|$$

$$(\frac{1}{3})^n \sin(\frac{\pi}{4} n) u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{\frac{1}{3\sqrt{2}} z}{(z - \frac{1}{3} e^{j\frac{\pi}{4}})(z - \frac{1}{3} e^{-j\frac{\pi}{4}})}, \qquad |z| > \frac{1}{3}$$

$$x[n] = \begin{cases} a^n & 0 \leq n \leq N-1, a > 0 \\ 0 & otherwise \end{cases} \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

$$z_k = ae^{2\pi k/N}, \; k = 0, 1, 2, \ldots, N-1$$

$$b^{|n|}, \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - bz^{-1}} - \frac{1}{1 - b^{-1} z^{-1}}, \qquad b < |z| < \frac{1}{b}, \quad 0 < b < 1$$

Examples

- If the $z-$transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is right sided, then the ROC is the region in the $z-$plane outside the outermost pole.
  Furthermore, if $x[n]$ is causal, then the ROC also includes $z = \infty$.
- If the $z-$transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is left sided, then the ROC is the region in the z-plane inside the innermost nonzero pole.
  Furthermore, if $x[n]$ is anticausal, then the ROC also includes $z = 0$.

Inverse $z-$Transform

$$x[n] = \frac{1}{2\pi j}\oint X(z) z^{n-1} dz$$

Example:

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \qquad |z| > \frac{1}{3}$$

$$X(z) = \frac{1}{(1 - \frac{1}{4}z^{-1})} + \frac{2}{(1 - \frac{1}{3}z^{-1})}$$

$$x[n] = \left(\frac{1}{4}\right) u[n] + 2\left(\frac{1}{3}\right) u[n]$$

Example

- $X(z) = \frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}, \qquad \frac{1}{4} < |z| < \frac{1}{3}$
  $x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$
- $X(z) = \frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}, \qquad |z| < \frac{1}{4}$
  $x[n] = -\left(\frac{1}{4}\right)^n u[-n-1] - 2\left(\frac{1}{3}\right)^n u[-n-1]$

Properties of $z-$Transform

- Linearity: $ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z)$
- Time Shifting: $x[n - n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z)$
- Scaling: $z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(\frac{z}{z_0})$, ROC $= |z_0|R$
- $x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(\frac{1}{z})$, ROC $= 1/R$
- $x_{(k)}[n] = \begin{cases} x[n/k] & if\ n = multiple\ of\ k \\ 0 & if\ n \neq multiple\ of\ k \end{cases} \overset{\mathcal{Z}}{\longleftrightarrow} X(z^k)$, ROC $= R^{1/k}$
- Conjugation: $x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$
- Convolution: $x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)X_2(z)$, ROC $R_1 \cap R_2$
- Differentiation in Time: $nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}$

Example

- $X(z) = \log(1 + az^{-1}),\ |z| > |a|$
  $-z\frac{d}{dz}X(z) = az^{-1}\frac{1}{(1+az^{-1})}$
  $nx[n] = a(-a)^{n-1}u[n-1] = -(-a)^n u[n-1]$
  $x[n] = \frac{-(-a)^n u[n-1]}{n}$
- $X(z) = \frac{az^{-1}}{(1-az^{-1})^2},\ |z| > |a|$
  $-zX(z) = \frac{d}{dz}\frac{1}{(1-az^{-1})} \overset{\mathcal{Z}}{\longleftrightarrow} na^n u[n]$

LTI Systems Characterized by Linear Difference Equations

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

If ROC $= |z| > 1$, $h[n] = (\frac{1}{2})^n u[n] + \frac{1}{3}(\frac{1}{2})^{n-1}u[n-1]$
If the system is anticausal, $h[n] = -(\frac{1}{2})^n u[-n-1] - \frac{1}{3}(\frac{1}{2})^{n-1}u[-n]$

LTI Systems Characterized by Linear Difference Equations

A second order system:

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

and its difference equation:

$$y[n] = \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$

The Unilateral $z-$Transform

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x[n] \stackrel{\mathcal{UZ}}{\longleftrightarrow} \mathcal{X}(z)$$

- $a^{n+1}u[n+1] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{z}{1-az^{-1}},\ |z| > a,$
  $a^{n+1}u[n+1] \stackrel{\mathcal{UZ}}{\longleftrightarrow} \frac{a}{1-az^{-1}},\ |z| > a,$
- $\mathcal{X}(z) = \frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{1}{(1-\frac{1}{4}z^{-1})} + \frac{2}{(1-\frac{1}{3}z^{-1})}$
  $x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n],\ n \geq 0,\ |z| > \frac{1}{3}$

Example

$y[n] + 3y[n-1] = x[n]$, $\mathcal{H}(z) = \frac{1}{1+3z^{-1}}$
Determine the output $y[n]$ to the input $x[n] = \alpha u[n]$

- $\mathcal{Y}(z) = \mathcal{H}(z)\mathcal{X}(z) = \frac{\alpha}{(1+3z^{-1})(1-z^{-1})} = \frac{3/4\alpha}{(1+3z^{-1})} + \frac{1/4\alpha}{(1-z^{-1})}$
  $y[n] = \alpha\left[\frac{1}{4} + \left(\frac{3}{4}\right)(-3)^n\right]u[n]$

Consider the same system with the inital conditions $y[-1] = \beta$.

- $\mathcal{Y}(z) + 3\beta + 3z^{-1}\mathcal{Y}(z) = \frac{\alpha}{1-z^{-1}}$
  $\mathcal{Y}(z) = -\frac{3\beta}{1+3z^{-1}} + \frac{\alpha}{(1+3z^{-1})(1-z^{-1})}$
  $$y[n] = \overbrace{-3\beta(-3)^n u[n]}^{Zero-input\ Solution} + \overbrace{\alpha\left[\frac{1}{4} + \left(\frac{3}{4}\right)(-3)^n\right]u[n]}^{Zero-state\ Solution}$$