Sampling and Interpolation

Lecture Notes

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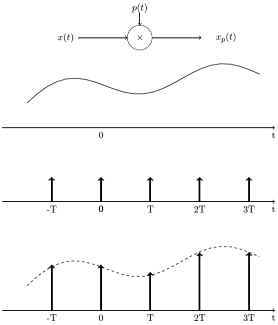
Some concepts and illustrations in this lecture are adapted from the textbooks,

Signals and Systems, 2nd Edition by Alan Oppenheim, Alan Willisky and H. Nawab, *Prentice Hall*.

Applied Digital Signal Processing, Dimitris G. Manolakis & K. Ingle, *Cambridge*.

Nyquist Sampling Theorem



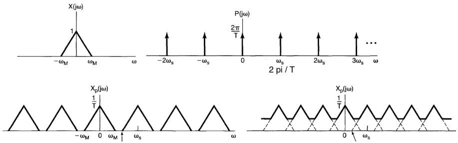
$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

$$x_p(t) = x_c(t)p(t)$$

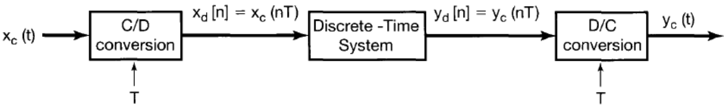
$$X_p(\omega) = \frac{1}{2\pi} X_c(\omega) * P(\omega)$$

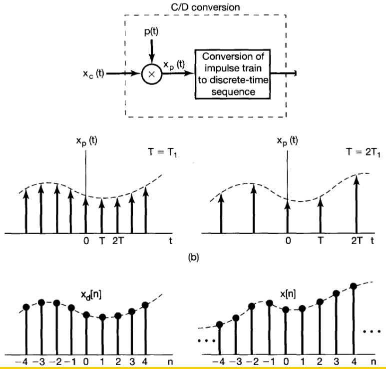
$$P(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\frac{2\pi}{T})$$

$$X_p(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(\omega - n\frac{2\pi}{T})$$



Analog to Digital Conversion





$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) \longleftrightarrow X_p(\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega nT}$$

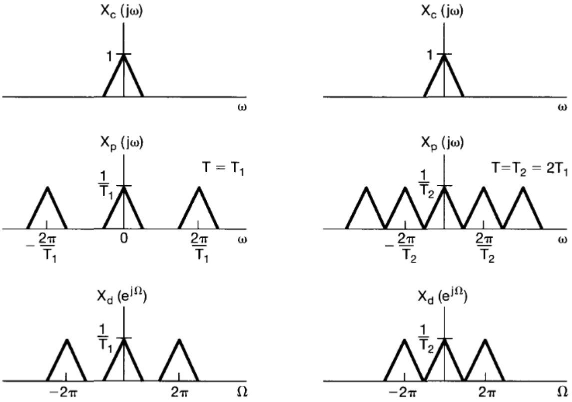
Discrete Fourier Transform of $X_d[n]$ is

$$X_d(\Omega) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_c[nT]e^{-j\Omega n}$$

Comparing $X_p(\omega)$ and $X_d(\Omega)$,

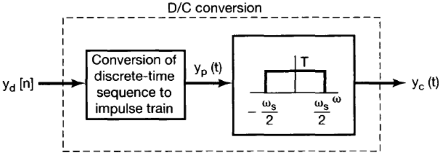
$$X_d(\Omega) = X_p(\omega)\Big|_{\omega=\frac{\Omega}{T}}$$

$$X_p(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(\omega - n\tfrac{2\pi}{T}) \text{ and } X_d(\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(\tfrac{\Omega}{T} - n\tfrac{2\pi}{T})$$



When $x_c(t)$ is converted into $x_d[n]$, the time axis is scaled by $1/T$ which leads to a scaling of frequency axis by T.

Reconstruction of a Signal from its Samples

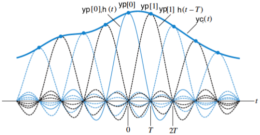


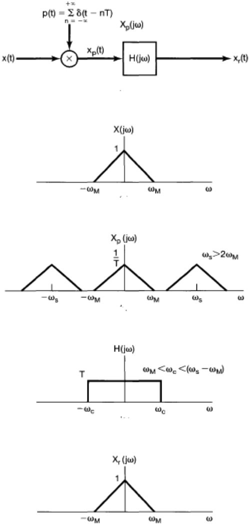
Since $\omega_s = \frac{2\pi}{T}$, $h(t) = sinc(\frac{t}{T})$

$$y_c(t) = y_p(t) * h(t)$$

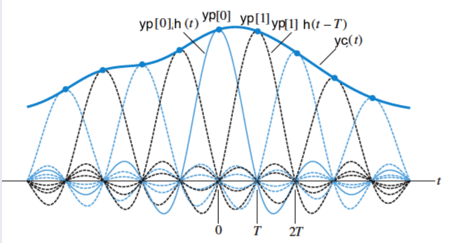
$$y_c(t) = \sum_{n=-\infty}^{\infty} y_p(nT)\delta(t - nT) * sinc(\frac{t}{T})$$

$$= \sum_{n=-\infty}^{\infty} y_p(nT) sinc(\frac{t-nT}{T})$$





Sinc Interpolation



$$y_c(t) = \sum_{n=-\infty}^{\infty} y_p[nT]h(t - nT) \qquad h(t) = sinc(\frac{t}{T})$$

```
y= [1 3 -3 5 8 3 5]';
x= [ 1 3 4 5 6 8 10]';
dx= 0.01; X_min = min(x)-1; X_max = max(x)+1;
X = linspace(X_min,X_max,1/dx)';
Y=sinc(repmat(X,1,length(x))-repmat(x',length(X),1)).* repmat(y',size(X,1),1);
plot(X,Y);grid;hold;stem(x,y);plot(X,sum(Y,2);
```

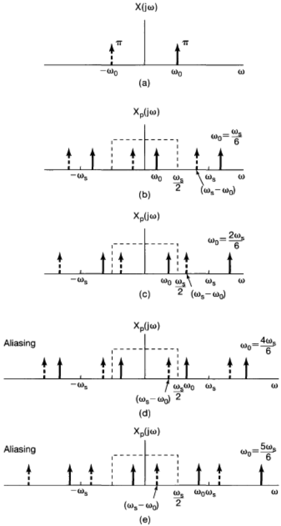
Example of Undersampling

$\omega_0 = \frac{\omega_s}{6}$, $x_r(t) = \cos(\omega_0 t) = x(t)$

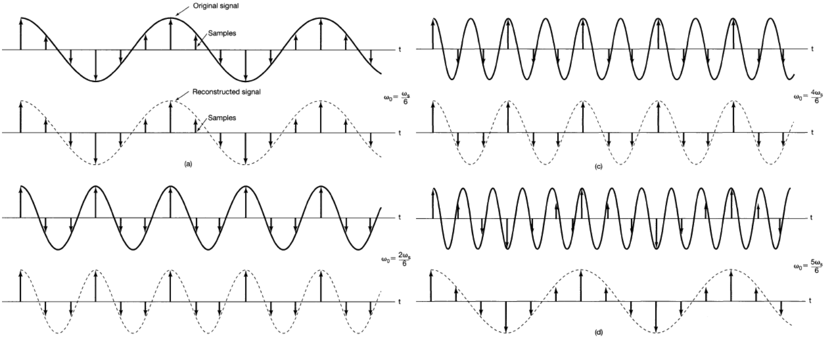
$\omega_0 = \frac{2\omega_s}{6}$, $x_r(t) = \cos(\omega_0 t) = x(t)$

$\omega_0 = \frac{4\omega_s}{6}$, $x_r(t) = \cos((\omega_s - \omega_t)) \neq x(t)$

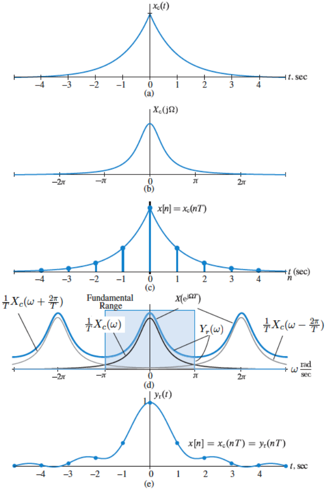
$\omega_0 = \frac{5\omega_s}{6}$, $x_r(t) = \cos((\omega_s - \omega_t)) \neq x(t)$



Effect of Undersampling: Aliasing



Aliasing in non-bandlimited signals



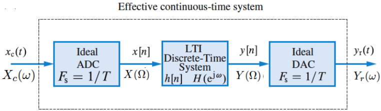
$$x_c(t) = e^{-A|t|} \overset{CTFT}{\longleftrightarrow} X_c(j\omega) = \frac{2A}{\omega^2+A^2}$$

$$x[n] = x_c(nT) = e^{-|A|nT} = (e^{-AT})^{|n|}$$

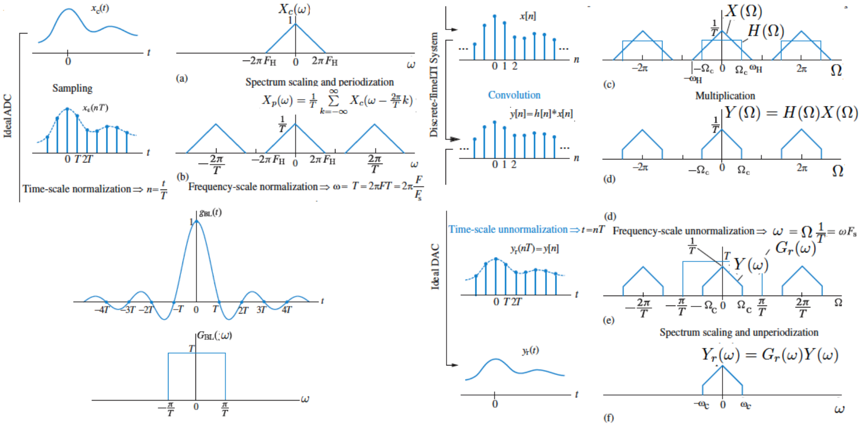
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \frac{1-a^2}{1-2a\cos\Omega+a^2}$$

$$X(\omega) = \frac{1-a^2}{1-2a\cos(\omega T)+a^2}$$

$$y_r(m\Delta t) \approx \sum_{N_1}^{N_2} x[n]\frac{\sin(\pi(m\Delta t-nT)/T)}{\pi(m\Delta-nT)/T}$$

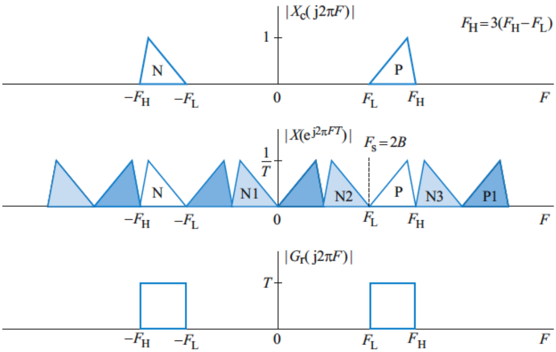


Discrete time filtering of Continuous Signals



$$g_r(t) = g_{BL}(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

Sampling of Bandpass Signals



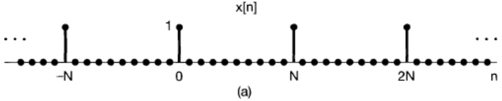
$$G_r(t) = \frac{\sin(\pi B t)}{\pi B t}\cos(2\pi F_C t)$$

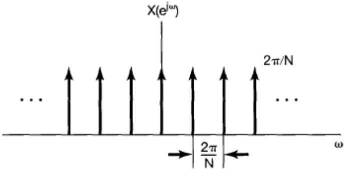
Sampling rate : $F_s = 2(F_H - F_L)$ if F_H is a multiple of $F_H - F_L$.

Center frequency : $F_C = (F_H + F_L)/2$

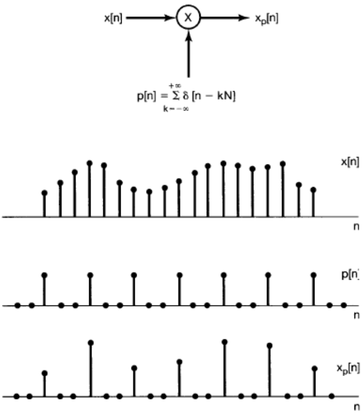
Example

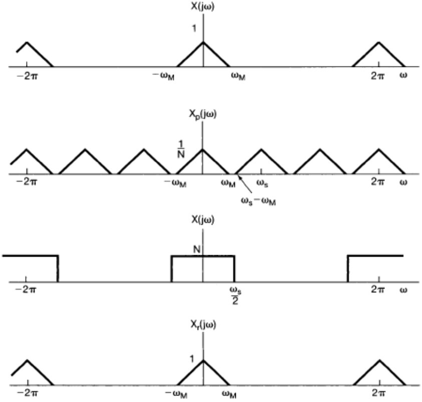
$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \qquad X(\omega) = \frac{1}{N} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \frac{2\pi k}{N})$$



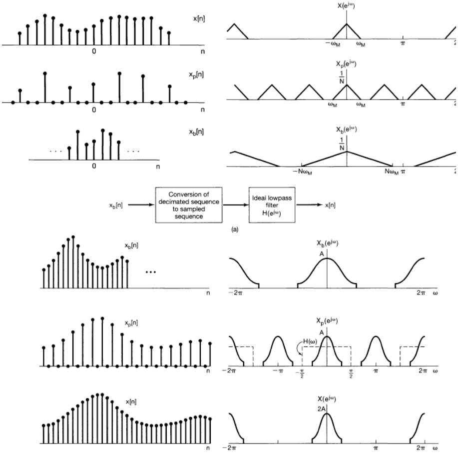


Sampling of Discrete Signals



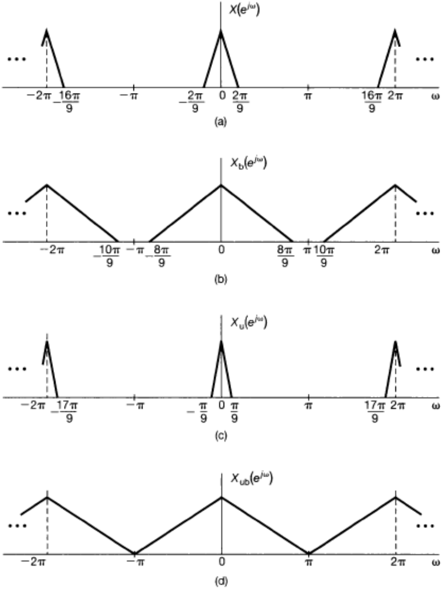


Discrete time decimation and interpolation



$$X_b(\omega) = \sum_{k=-\infty}^{\infty} x_p[kN]e^{-j\omega}$$

$$X_b(\omega) = X_p(\omega/N)$$



Original spectrum

Spectrum after downsampling by 4

Spectrum after upsampling $x[n]$ by 2.

Spectrum after upsampling $x[n]$ by 2 then downsampling by 9.