Hierarchical Models and Random Effects

Lecture Notes

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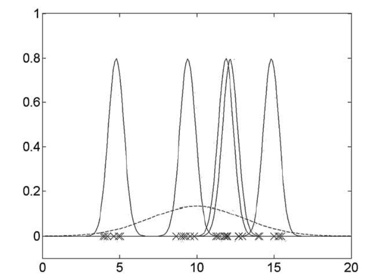
Some concepts and illustrations in this lecture are adapted from the textbook,
Statistical Parametric Mapping: The Analysis of Functional Brain Images, Editors: K. Friston, J. Ashburner, S. Kiebel, T. Nichols and W. Penny, *Academic Press*, 2006.

Effects from N subjects with n replications per subject, the population effect is modelled by a two-level process:

$$y_{ij} = w_i + e_{ij}$$
$$w_i = w_{pop} + z_i$$

w_i : true mean effect for subject i
e_{ij} : within subject error $\sim \mathcal{N}(0, \sigma_w^2)$
y_{ij} : j^{th} observed effect for subject i
$z_i \sim \mathcal{N}(0, \sigma_b^2)$ between-subject error for the i^{th} subject



Collapsed Model

$y_{ij} = w_{pop} + z_i + e_{ij}$

Maximum Likelihood Estimate :

$$\hat{w}_{pop} = \frac{1}{Nn} \sum_{i=1}^{N} \sum_{j=1}^{n} y_{ij}$$

$$Var[\hat{w}_{pop}] = \frac{\sigma_b^2}{N} + \frac{\sigma_w^2}{Nn}$$

Simpler Approach than 2 level Model

Summary Statistics Approach

$$\begin{aligned}\bar{w}_i &= w_i + e_i \\ w_i &= w_{pop} + z_i\end{aligned}$$

w_i : true mean effect for subject i
$\bar{w}_i$: sample mean effect for subject i
w_{pop} : true effect for the population
Simpler because it is based on sample mean, $\bar{w}_i$ rather than all samples y_{ij}. $\mathrm{Var}(e_i) = \sigma_w^2/n$
$\mathrm{Var}(z_i) = \sigma_b^2 \longrightarrow \bar{w}_i = w_{pop} + z_i + e_i$
Population Mean : $\hat{w}_{pop} = \frac{1}{N}\sum_{i=1}^{N} \bar{w}_i \longrightarrow \mathrm{E}(\hat{w}_{pop}) = w_{pop}$
$\mathrm{Var}(\hat{w}_{pop}) = \frac{\sigma_b^2}{N} + \frac{\sigma_w^2}{Nn}$

FIXED EFFECT ANALYSIS

Single Level Model

$y_{ij} = w_i + e_{ij}$

Subject mean $\hat{w}_i = \frac{1}{n}\sum_{j=1}^{n} y_{ij} \longrightarrow \mathrm{Var}(\hat{w}_i) = \frac{\sigma_w^2}{n}$

Group mean $\hat{w}_{pop} = \frac{1}{N}\sum_{i=1}^{N} \hat{w}_i \longrightarrow \mathrm{Var}(\hat{w}_{pop}) = \frac{\sigma_w^2}{Nn}$

Partitioned Error Model for r 3×3 ANOVA

1^{st} Stage :

The differential effects for each subject are computed. For each subject n, the full model

$$\rho_{nq} = \tau_1^A + \tau_2^A + e_{nq}$$

is compared with the reduced model $\rho_{nq} = e_{nq}$ where the null hypothesis is $\mathcal{H}_0 : \tau_1^A = \tau_2^A = 0$.

The $(K_1 - 1)(K_2 - 1)$ differential effects ρ_{nqr} are computed and the full model

$$\rho_{nqr} = \tau_{11}^{AB} + \tau_{12}^{AB} + \tau_{21}^{AB} + \tau_{22}^{AB} + e_{nqr}$$

is compared with the reduced model $\rho_{nqr} = e_{nqr}$ where the null hypothesis is $\mathcal{H}_0 : \tau_{qr}^{AB} = 0$ for all $q, r \in \{1, 2\}$.

Partitioned Error Model foANOVA

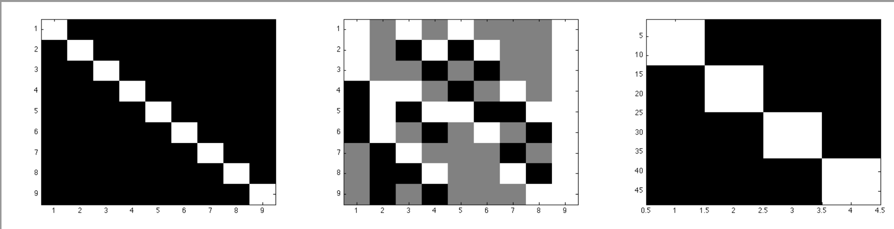
2^{nd} Stage:

1. If the $F-$contrasts have R_c rows for each subject, we have $R_c \times N$ paremeters to be modeled.
2. Set up a design matrix $\mathbf{X_2} = \mathbf{I}_{R_c} \otimes \mathbf{1}_N$
3. Using the $F-$contrast $C_2 = \mathbf{I}_{R_c}$ find the orthogonal contrast $C_0 = \mathbf{I}_p - C_2 C_2^-$ and the reduced model design matrix $\mathbf{X_{2}}_0 = \mathbf{X}_2 C_0$
4. Fit the second level model and test for the effect.

A partitioned error example: A 3×3 ANOVA with N=12

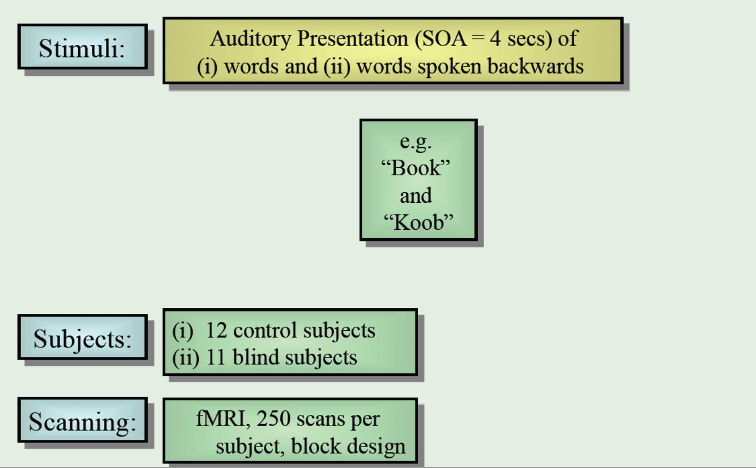
The first level design matrix $\mathbf{X}_r$ is obtained by rotating $\mathbf{X}$ with $\mathbf{C}^T$

$$\mathbf{C}^T = [\mathbf{D}_1 \otimes \mathbf{C}_2, \mathbf{C}_1 \otimes \mathbf{D}_2, \mathbf{D}_1 \otimes \mathbf{D}_2]$$



columns $1-2$: the main effect of A,
columns $3-4$: the main effect of B,
columns $5-8$: the interaction.
In the second stage, to test an effect i.e. the interaction, the design matrix $\mathbf{X}_2$ is set up as $\mathbf{I}_4 \otimes \mathbf{1}_{12}$ and the $F-$contrast $\mathbf{C}_2 = \mathbf{I}_4$.

Example : Noppeney et al. *Brain* (2003), 126, 1620-1627



Click two-choice key press if YES/NO

- Sound : Is it usually quiet/loud?, Bark, Bang, Siren, Whisper
- Visual : Is it always curved?, Angle, Cone, Pyramid, Oval, Is it usually dark?, Brown, Dusk, Glow, Flash
- Hand action : Is it a hand action with/without a tool?, Chisel, Knit, Tapping, Tickle
- Motion : Is it a jumping movement?, Leap, Swimming, Climb, Springing, Is it a slow movement?, Tiptoeing, Dawdle, Gallop, Run

Decide whether it is male voice record?
For each semantic condition, there was a matched baseline condition that employed the same recorded stimuli after digital reversal, to remove lexical and semantic content.

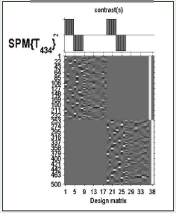
The conditions were modelled in an event-related (ER) fashion:
Regressors:

1. ER pulses convolved with HRF (4 task conditions)
2. Realignment parameters (6)
3. Reaction times for all activation and control conditions (4 task + 4 baseline)
4. Mean effect (1)

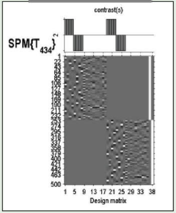
In addition to modelling each condition, the statistical model included instructions and errors. Nuisance covariates included the realignment parameters (to account for motion artefacts) and reaction times that were modelled in an eventrelated fashion separately for (i) all activation conditions and (ii) all control conditions (to account for differences in reaction times across conditions).

1st level:

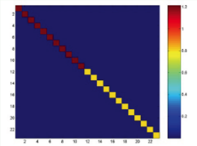
Controls



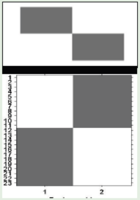
Blinds



2nd level:

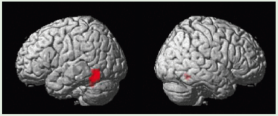


V



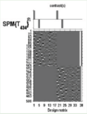
$c^T = [1 \quad -1]$

X



ANOVA

1st level: Motion, Sound, Visual, Action



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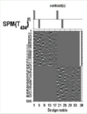

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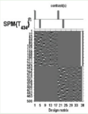

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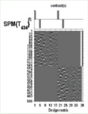
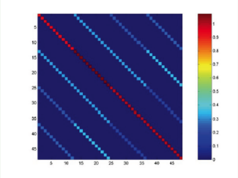
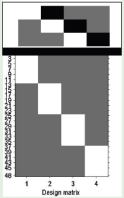


2nd level:

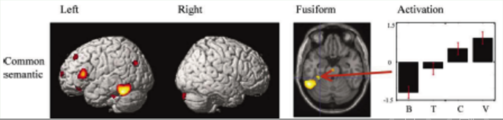


V



X

$$c^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$



1×4 ANOVA

$$C_2 = 1 \text{ and } D_1 = -\mathit{diff}(\mathbf{I_4})^T = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The contrast C for the main effect of semantic condition

$$C = D_1 \otimes C_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$