BASIC AXIOMS OF PROBABILITY

Lecture Notes

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Combinatorics

Roll two dice. What is the most likely sum?

Outcomes are ordered pairs $(i, j), 1 \leq i \leq 6, 1 \leq j \leq 6$.

sum	no. of outcomes
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1

Our answer is 7, and $P(sum = 7) = \frac{6}{36} = \frac{1}{6}$

Roll a die 4 times. What is the probability that you get different numbers?

Step 1: Identify the set of equally likely outcomes. In this case, this is the set of all ordered 4−tuples of numbers $1, \ldots, 6$. That is, $\{(a, b, c, d) : a, b, c, d \in \{1, \ldots, 6\}\}$

Step 2: Compute the number of outcomes. In this case, it is 6^4.

Compute the number of good outcomes. In this case it is $6 \cdot 5 \cdot 4 \cdot 3$.

The answer then $\frac{6\cdot5\cdot4\cdot3}{6^4} = \frac{5}{18}$.

Flemish aristocrat and notorious gambler Chevalier de Mere (1654) used to bet even money that he would get at least one 6 in four rolls of a fair die. The probability of this is 4 times the probability of getting a 6 in a single die, i.e., $4/6 = 2/3$; clearly he had an advantage and indeed he was making money. Now he bets even money that within 24 rolls of two dice he gets at least one double 6. This has the same advantage ($24/6^2 = 2/3$), but now he is losing money. Why?

In game 1, there are 4 rolls and he wins with at least one 6. The number of good events is $6^4 - 5^4$, as the number of bad events is 5^4. Therefore

$P(win) = 1 - \left(\frac{5}{6}\right)^4 \approx 0.5177.$

In Game 2, there are 24 rolls of two dice and he wins by at least one pair of 6's rolled. The number of outcomes is 36^{24}, the number of bad ones is 35^{24}, thus the number of good outcomes equals $36^{24} - 35^{24}$. Therefore,

$P(win) = 1 - \left(\frac{35}{36}\right)^{24} \approx 0.4914$

One should also note that both probabilities are barely different from 0.5, so de Mere was gambling a lot to be able to notice the difference.

Permutations

Assume you have n objects. The number of ways to fill n ordered slots with them is $n(n-1)\cdots 2 \cdot 1 = n!$, while the number of ways to fill $k \leq n$ ordered slots is $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$

Shuffle a deck of cards.

A What is the probability that the top card is an *Ace*?
$P(A) = \frac{4 \cdot 51!}{52!} = \frac{1}{13}$

B What is the probability that all cards of the same suit end up next to each other?
$P(B) = \frac{4! \cdot 13! \cdot 13! \cdot 13! \cdot 13!}{52!} \approx 4.5 \cdot 10^{-28}$ which can almost never happen in practice.

C What is the probability that hearts are together?
$P(C) = \frac{13! \cdot 40!}{52!} = 6 \cdot 10^{-11}$

A bag has 6 pieces of paper, each with one of the letters, E, E, P, P, P, and R, on it. Pull 6 pieces at random out of the bag (1) without, and (2) with replacement. What is the probability that these pieces, in order, spell *PEPPER*?

(1) An outcome is an ordering $E_1 E_2 P_1 P_2 P_3 R$
These can be ordered in 6! ways if all events are without replacement.
Good Events : $3! \cdot 2! \cdot 1!$
$\text{P} = \frac{3! \cdot 2! \cdot 1!}{6!} = \frac{1}{60}$
(2) All events with replacement : 6^6
Good Events : $3^3 \cdot 2^2 \cdot 1^1$
$\frac{3^3 \cdot 2^2 \cdot 1}{6^6} = \frac{1}{2 \cdot 6^3}$

Sit 3 men and 3 women at random (1) in a row of chairs and (2) around a table. Compute P(all women sit together) . In case (2), also compute P(men and women alternate).

(1) All possible events : $6!$
Number of good events : $4! \cdot 3!$
$P = \frac{4! \cdot 3!}{6!} = \frac{1}{5}$
(2) All possible events : $5!$ (fix one person i.e. a man and place others around)
Number of good events : $3! \cdot 3!$
$P = \frac{3! \cdot 3!}{5!} = \frac{3}{10}$
Number of good events for alternate placing : $3! \cdot 2!$ (women's placing options · men's placing options)
$P = \frac{3! \cdot 2!}{5!} = \frac{1}{10}$

Combinations

Let $\binom{n}{k}$ be the number of different subsets with k elements of a set with n elements. Then,

$$\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

The number of ways to divide a set of n elements into r (distinguishable) subsets of $n_1, n_2, \ldots, n_r$ elements, where $n_1 + \cdots + n_r = n$, is denoted by $\binom{n}{n_1 \ldots n_r}$ and

$$\binom{n}{n_1 \ldots n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\ldots\binom{n-n_1-\cdots-n_{r-1}}{n_r} = \left(\frac{n!}{n_1!n_2!\ldots n_r!}\right)$$

A fair coin is tossed 10 times. What is the probability that we get exactly 5 *Heads*?

All events : 2^{10}

Good events : $\binom{10}{5}$

$$P = \frac{\binom{10}{5}}{2^{10}} \approx 0.2461$$

We have a bag that contains 100 balls, 50 of them red and 50 blue. Select 5 balls at random. What is the probability that 3 are blue and 2 are red?

All events : $\binom{100}{5}$

Good events : $\binom{50}{3}\binom{50}{2}$

$$P = \frac{\binom{50}{3}\binom{50}{2}}{\binom{100}{5}} \approx 0.3189$$

Repeat : Shuffle a standard deck of 52 cards and deal 13 cards to each of the 4 players. What is the probability that each player gets an *Ace*?

All events : 52! different orderings

Good events : $(13)^4 \cdot 4! \cdot 48!$ (Each Ace has 13 slots)(Aces have 4! different orderings)(The rest of the cards have 48! different orderings)

$P = \frac{13^4 \cdot 4! \cdot 48!}{52!}$

Alternatively,

All events : $\binom{52}{4}$

Good events : 13^4

$$P = \frac{13^4}{\binom{52}{4}}$$

What is the probability that one person has all four Aces?

All events : $\binom{52}{4}$ Good events :

$$\frac{\binom{4}{1}\binom{13}{4}}{\binom{52}{4}} 48! = \frac{4(13x12x11x10)48!}{52!} = 0.011$$

Roll a die 12 times. What is the probability that each number appears exactly twice?

All events : 6^{12}

Good events : $\begin{pmatrix} 12 \\ 2 \end{pmatrix}\begin{pmatrix} 10 \\ 2 \end{pmatrix}\cdots\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

- $P = \frac{\begin{pmatrix} 12 \\ 2 \end{pmatrix}\begin{pmatrix} 10 \\ 2 \end{pmatrix}\cdots\begin{pmatrix} 2 \\ 2 \end{pmatrix}}{6^{12}}$

What is the probability that 1 appears exactly 3 times, 2 appears exactly 2 times ?

Good events : $\begin{pmatrix} 12 \\ 3 \end{pmatrix}\begin{pmatrix} 9 \\ 2 \end{pmatrix}4^7$

- $P = \frac{\begin{pmatrix} 12 \\ 3 \end{pmatrix}\begin{pmatrix} 9 \\ 2 \end{pmatrix}4^7}{6^{12}}$

A middle row on a plane seats 7 people. Three of them order chicken (C) and the remaining four pasta (P). The flight attendant returns with the meals, but has forgot who ordered what and discovers that they are all asleep, so she puts the meals in front of them at random. What is the probability that they all receive correct meals?

All events : $\frac{7!}{3!\cdot 4!} = \binom{7}{3} = 35$ (Possible orderings of 3 (C) and 4 (P))

Good events : 1 (A single case where each seat receives the correct meal.)

$P = \frac{1}{35}$

What is the probability that no one who ordered C gets C?

Good events : $\frac{(4\cdot 3\cdot 2)}{3!}$ (Ordering of (C) in (P) seats) , $P = \frac{\frac{(4\cdot 3\cdot 2)}{3!}}{35} = \frac{4}{35}$

What is the probability that a single person who ordered C gets C?

Good events : $\binom{3}{1}\binom{4}{2}$, $P = \frac{\binom{3}{1}\binom{4}{2}}{35} = \frac{18}{35}$

What is the probability that two persons who ordered C gets C?

Good events : $\binom{3}{2}\binom{4}{1}$, $P = \frac{\binom{3}{2}\binom{4}{1}}{35} = \frac{12}{35}$

Axioms of Probability

A probability space is a triple $(\Omega, \mathcal{F}, P)$

- Ω : set of outcomes called the sample space.
- $\mathcal{F}$: a collection of events which is set of subsets of Ω.
- P : a number attached to every event $A \in \mathcal{F}$ and satisfies the following three axioms:

Axiom 1 : For every event A, $P(A) \geq 0$.

Axiom 2 : $P(\Omega) = 1$.

Axiom 3 : If $A_1, A_2, \ldots$ is a sequence of pairwise disjoint events, then

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)..$$

Consequences of the axioms

1. $P(\varnothing) = 0$.
2. If $A_1 \cap A_2 = \varnothing$, then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.
3. $P(A^c) = 1 - P(A)$.
4. $0 \leq P(A) \leq 1$.
5. If $A \subset B, P(B) = P(A) + P(B \setminus A) \geq P(A)$.
6. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
7. $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$.
   Generally,
   $$P(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i \leq j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i \leq j \leq k \leq n} P(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n-1} P(A_1 \cap \cdots \cap A_n).$$

Pick an integer in $[1, 1000]$ at random. Compute the probability that it is divisible neither by 12 nor by 15.

i Numbers divisible by 12 : $A = \frac{1000}{12} = 83.$

ii Numbers divisible by 15 : $B = \frac{1000}{15} = 66.$

iii LCM : least common multiplier $LCM(12, 15) = 2^2 \cdot 3 \cdot 5 = 60$
Numbers both divisible by 12 and 15 :
$A \cap B = \frac{1000}{LCM(12,15)} = \frac{1000}{60} = 16.$
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{83}{1000} + \frac{66}{1000} - \frac{16}{1000} = \frac{133}{1000}$

- $P = 1 - P(A \cup B) = 1 - \frac{133}{1000} = \frac{867}{1000} = 0.867$

Sit 3 men and 4 women at random in a row. What is the probability that either all the men or all the women end up sitting together?

- All orderings 7!

  A: All men sitting together : $(1+4)! \cdot 3!$

  B : All women sitting together : $(1+3)! \cdot 4!$

  $A \cap B$: Both men and women sitting together : $3! \cdot 4! \cdot 2!$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5!3!}{7!} + \frac{4!4!}{7!} - \frac{3!4!2!}{7!} = \frac{12}{35}$

A group of 3 Norwegians, 4 Swedes, and 5 Finns is seated at random around a table. Compute the probability that at least one of the three groups ends up sitting together.

Let's put a Norwegian on a single spot then all possible orderings $= (12-1)! = 11!$

A : Norwegians sitting together : $9! \cdot 3!$

B : Swedes sitting together : $8! \cdot 4!$

C : Finns sitting together : $7! \cdot 5!$

$A \cap B$: Both Norwegians and Swedes sitting together: $6! \cdot 4! \cdot 3!$

$A \cap C$: Both Norwegians and Finns sitting together : $5! \cdot 5! \cdot 3!$

$B \cap C$: Both Swedes and Finns sitting together : $4! \cdot 5! \cdot 4!$

$A \cap B \cap C$: All three groups sitting together : $2! \cdot 4! \cdot 5! \cdot 3!$

- $P(A \cup B \cup C) = \frac{9!3!+8!4!+7!5!-6!4!3!-5!5!3!-4!5!4!+2!4!5!3!}{11!} = 0.09.$

Matching problem. A large company with n employees has a scheme according to which each employee buys a Christmas gift and the gifts are then distributed at random to the employees. What is the probability that someone gets his or her own gift?

A_i : An employee has a chance of $\frac{1}{n}$ to recieve his/her own gift.

$A_i \cap A_j$: 2 employees have a chance of $\frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$ to recieve their own gift.

$A_i \cap A_j \cap A_k$: 3 employees have a chance of $\frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}$ to recieve their own gift.

...

$A_1 \cap \cdots \cap A_n$: All employees to receive their own gift has a chance of $\frac{1}{n!}$

$P(A_1 \cup \cdots \cup A_n) = n \cdot \frac{1}{n} - \binom{n}{2} \cdot \frac{1}{n(n-1)} + \binom{n}{3} \cdot \frac{1}{n(n-1)(n-2)} - \cdots + (-1)^{n-1} \frac{1}{n!}$
$= 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{n-1} \frac{1}{n!}$
Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \frac{1}{2!} - \frac{1}{3!} + \ldots$

- $P(A_1 \cup \cdots \cup A_n) = 1 - e^{-1} \approx 0.6321$ (as $n \to \infty$)

Birthday Problem. Assume that there are k people in the room. What is the probability that there are 2 who share a birthday?

Ignoring leap years, assuming all birthdays are equally likely, from n possible birthdays, we sample k times with replacement.

All events : n^k (k people to be placed on n slots).

Bad events : $n \cdot (n-1) \dots (n-k+1)$ (placing each person to a different slot).

$P(Bad) = \frac{n \cdot (n-1) \cdots (n-k+1)}{n^k}$, $P = 1 - P(Bad)$

Roll a die 12 times. Compute the probability that a number occurs 6 times and two other numbers occur three times each.

All events : 6^{12}

$i = \{1, 2, \dots, 6\}$ can be chosen $\binom{6}{1}$ different ways and each can have $\binom{12}{6}$ possible placements. Two other numbers can be chosen in $\binom{5}{2}$ different ways each of which can be placed in $\binom{6}{3}$ different ways.

- $P = \dfrac{\binom{6}{1} \cdot \binom{5}{2} \cdot \binom{12}{6} \cdot \binom{6}{3} \cdot \binom{3}{3}}{6^{12}}$

Poker Hands. The word *value* refers to $A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2$. This sequence orders the cards in descending consecutive values, with one exception: an *Ace* may be regarded as 1 for the purposes of making a straight (but note that, for example, $K, A, 1, 2, 3$ is not a valid straight sequence —A can only begin or end a straight).

All events : $\binom{52}{5}$, $P = \frac{\textit{Good events}}{\textit{All events}}$

What is the probability of *one pair* : two cards of the same value plus 3 cards with different values?

$J\spadesuit \cdot J\clubsuit \cdot 9\heartsuit \cdot Q\clubsuit \cdot 4\spadesuit$

Good events : $\binom{4}{2}\binom{13}{1}\binom{12}{3}4^3$.

What is the probability of *two pairs*: two pairs plus another card of different value ?

$J\spadesuit \cdot J\clubsuit \cdot 9\heartsuit \cdot 9\clubsuit \cdot 3\spadesuit$

Good events : $\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$

What is the probability of a *straight* : five cards with consecutive values ?

$5\heartsuit \cdot 4\clubsuit \cdot 3\clubsuit \cdot 2\heartsuit \cdot A\spadesuit$

Good events : $10 \cdot 4^5$

What is the probability of a *flush*: five cards of the same suit ?

$K\clubsuit \cdot 9\clubsuit \cdot 7\clubsuit \cdot 6\clubsuit \cdot 3\clubsuit$

Good events : $\binom{13}{5}\binom{4}{1}$

What is the probability of a *full house*: a three of a kind and a pair ?

$J\clubsuit \cdot J\diamondsuit \cdot J\heartsuit \cdot 3\clubsuit \cdot 3\spadesuit$

Good events : $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$

What is the probability of *four of a kind*: four cards of the same value ?

$K\clubsuit \cdot K\diamondsuit \cdot K\heartsuit \cdot K\spadesuit \cdot 10\spadesuit$

Good events : $\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}$

What is the probability of a *straight flush*: five cards of the same suit with consecutive values ?

$A\spadesuit \cdot K\spadesuit \cdot Q\spadesuit \cdot J\spadesuit \cdot 10\spadesuit$

$$\text{Good events}: \binom{4}{1} \cdot 10$$

What is the probability of not getting any hand listed above ?

$$P = P(\textit{All cards with different values}) - P(\textit{straight or flush})$$
$$= \frac{\binom{13}{5} \cdot 4^5 - \left[10 \cdot 4^5 + \binom{13}{5}\binom{4}{1} - \binom{4}{1} \cdot 10\right]}{\binom{52}{5}}$$

Assume that 10 Finns, and 10 Danes, are to be distributed at random into 10 rooms, 2 per room. What is the probability that exactly $2i$ rooms are mixed, $i = 0, \ldots, 5$?

All events : $\binom{20}{10}$,

Good Events : $\binom{10}{2i} 2^{2i} \binom{10-2i}{5-i}$

(mixed rooms) (mixed orderings) (same national rooms)