Power Spectrum Estimation

Lecture Notes

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Some concepts and illustrations in this lecture are adapted from the textbook,
Statistical Digital Signal Processing and Modeling, Monson Hayes, *Wiley*.

Nonparametric Power Spectrum Methods

Nonparametric Power Spectrum Methods

1. Periodogram
2. Modified Periodogram
3. Bartlett's Method
4. Welch's Method
5. Blackman-Tukey Method
6. Multitaper Method
7. Minimum Variance Spectrum Estimation
8. Maximum Entropy Method

Periodogram

Power spectrum of a WS stationary process $x(n)$ is

$$P_z(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_x e^{jwk}$$

For an *autoregression ergodic* process

$$r_x(k) = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+k)x^*(n)$$

Since $x(n)$ is finite *i.e.* $\{x(0), x(1), \ldots, x(N-1)\}$

$$\hat{r}_x(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n+k)x^*(n) = \frac{1}{N} \sum_{n=0}^{N-1-k} x(n+k)x^*(n), \quad k \in [0, N-1]$$

$\hat{r}_x(-k) = \hat{r}_x^*(k)$

$$\hat{P}_{per}(e^{j\omega}) = \sum_{k=-N+1}^{N-1} \hat{r}_x(k) e^{-j\omega k}$$

$$x_N(n) = \begin{cases} x(n) & ; \quad 0 \le n < N, \\ 0 & ; \quad otherwise \end{cases}$$

$x_N(n) = w_R(n)x(n)$

$$\hat{r}_x(k) = \frac{1}{N}\sum_{n=-\infty}^{\infty} x_N(n+k)x_N^*(n) = \frac{1}{N}x_N(k) * x_N^*(-k)$$

$$\hat{P}_{per}(e^{j\omega}) = \mathcal{F}\left\{\hat{r}_x(k)\right\} = \frac{1}{N}X_N(e^{j\omega})X_N^*(e^{j\omega}) = \frac{1}{N}\left|X_N(e^{j\omega})\right|^2$$

$$X_N(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_N(n)e^{-j\omega n} = \sum_{n=0}^{N-1} x_N(n)e^{-j\omega n}$$

Periodogram of white noise; $r_x(k) = \sigma_x^2\delta(k)$

$$P_{per}(e^{j\omega}) = \sigma_x^2$$

Performance of Periodogram

Mean square convergence: $\lim_{N\to\infty} \left\{ \left[\hat{P}_{per}(e^{j\omega}) - P_x(e^{j\omega}) \right] \right\} = 0$
means

1. Asymptotically unbiased : $\lim_{N\to\infty} \left\{ \hat{P}_{per}(e^{j\omega}) \right\} = P_{per}(e^{j\omega})$
2. Consistent : $\lim_{N\to\infty} Var \left\{ \hat{P}_{per}(e^{j\omega}) \right\} = 0$

$$E\{\hat{r}_x(k)\} = \frac{1}{N} \sum_{n=-\infty}^{\infty} E[x_N(n+k)x_N^*(n)] = \frac{1}{N} \sum_{n=0}^{N-1-k} r_x(k) = \frac{N-k}{N} r_x(k)$$

$$E\left\{\hat{r}_x(k)\right\} = w_B(k) r_x(k)$$

$$w_B(k) = x(t) = \begin{cases} \frac{N-|k|}{N} & : \quad |k| \leq N, \\ 0 & : \quad k > N \end{cases}$$

$$E\left\{ \hat{P}_{per}(e^{j\omega}) \right\} = E\left\{ \sum_{k=-N+1}^{N-1} \hat{r}_x(k) e^{-j\omega k} \right\} = \sum_{k=-N+1}^{N-1} r_x(k) w_B(k) e^{-j\omega k}$$

$E\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega})$ and $W_B(e^{j\omega}) = \frac{1}{N}\left[\frac{\sin(N\omega/2)}{\sin(\omega/2)}\right]^2$

$\hat{P}_{per}(e^{j\omega})$ is biased but it is asymptotically unbiased :

$\lim_{N\to\infty}\left\{\hat{P}_{per}(e^{j\omega})\right\} = P_{per}(e^{j\omega})$

Periodogram of sinusiods in noise:

$x(n) = A\sin(n\omega_1 + \phi_1) + A\sin(n\omega_2 + \phi_2) + v(n)$

$\hat{P}_x(e^{j\omega}) = \sigma_v^2 + \frac{1}{2}\pi A^2[\delta(\omega - \omega_1) + \delta(\omega - \omega_1)] + \frac{1}{2}\pi A^2[\delta(\omega - \omega_2] + \delta(\omega - \omega_2))$

Resolution of Periodogram

$$Res[\hat{P}_{per}(e^{j\omega})] = 0.89\frac{2\pi}{N} < \Delta\omega = |\omega_1 - \omega_2|$$

Variance of Periodogram

$$Var\left\{\hat{P}_{per}(e^{j\omega})\right\} = P_x^2(e^{j\omega})$$

Periodogram is not a consistent estimate.

Modified Periodogram

$$\hat{P}_M(e^{j\omega}) = \frac{1}{NU}\left|\sum_{n=-\infty}^{\infty} x(n)w_(n)e^{-j\omega n}\right|^2$$

$$U = \frac{1}{N}\sum_{n=0}^{N-1}|w(n)|^2$$

Bias

$$E\{\hat{P}_{per}(e^{j\omega})\} = \frac{1}{2\pi NU}P_x(e^{j\omega}) * |W(e^{j\omega})|^2$$

Variance

$$Var\left\{\hat{P}_M(e^{j\omega})\right\} \approx P_x(e^{j\omega})^2$$

Resolution

Rectangular	$0.892\pi/N(-13dB)$
Bartlett	$1.282\pi/N(-27dB)$
Hanning	$1.442\pi/N(-32dB)$
Hamming	$1.302\pi/N(-43dB)$
Blackman	$1.682\pi/N(-58dB)$

Bartlett's Method: Periodogram Averaging

Let $x_i(n)$ are uncorrelated realizations of $x(n)$ over $0 \leq n < L$ with $i = 1, 2 \ldots, K$.

$$\hat{P}_{per}^{(i)}(e^{j\omega}) = \frac{1}{L}\left|\sum_{n=0}^{L-1} x_i(n)e^{-j\omega n}\right|^2 \quad ; \quad i = 1, 2, \ldots, K$$

$$\hat{P}_{per}(e^{j\omega}) = \frac{1}{K}\sum_{i=1}^{K} \hat{P}_{per}^{(i)}(e^{j\omega})$$

$$E\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{K}\sum_{i=1}^{K} \hat{P}_{per}^{(i)}(e^{j\omega}) = \frac{1}{2\pi} P_x(e^{j\omega}) W_B(e^{j\omega})$$

$\hat{P}_{per}(e^{j\omega})$ is asymptotically unbiased.
Assuming that data segments are uncorrelated;

$$Var\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{K^2}\sum_{i=1}^{K} Var\left\{\hat{P}_{per}^{(i)}(e^{j\omega})\right\} \approx \frac{1}{K} P_x(e^{j\omega})$$

If K and L go to infinity $\hat{P}_B(e^{j\omega})$ will be a consistent estimate of $P_x(e^{j\omega})$.
Resolution : $0.892\pi/L = (0.89K)2\pi/N$ indicating that it is K times poorer.

Welch's Method: Averaging Modified Periodograms

$x_i(n)$ overlaps by offsets of D points.

$x_i(n) = x(n + iD)$; $n = 0, 1, \ldots, L - 1$ with an amount of overlap $L - D$ points.

$N = L + D(K - 1)$ if segments cover the entire data points.

No overlap means $K = N/L$ segments.

%50 overlap means $D = L/2$ and $K = 2\left(\frac{N}{L}\right) - 1$ with the same resolution as no overlap but doubling the number of segments averaged.

$$\hat{P}_W(e^{j\omega}) = \frac{1}{KLU} \sum_{i=0}^{K-1} \left| \sum_{n=0}^{L-1} w(n)x(n + iD)e^{-j\omega n} \right|^2$$

$$\hat{P}_W(e^{j\omega}) = \frac{1}{K} \sum_{i=0}^{K-1} \hat{P}_M^{(i)}(e^{j\omega})$$

$$E\{\hat{P}_W(e^{j\omega})\} = E\{\hat{P}_M(e^{j\omega})\} = \frac{1}{2\pi LU} P_x(e^{j\omega}) * \left| W^{j\omega}) \right|^2$$

It is asymptotically unbiased.

Resolution is window dependent.

$Var\left\{\hat{P}_W(e^{j\omega})\right\} \approx \frac{9}{16}\frac{L}{N}P_x^2(e^{j\omega})$

Blackman-Tukey Method: Perodogram Smoothing

$\hat{r}_x(N-1) = \frac{1}{N}x(N-1)x(0)$: unreliable estimate

$$P_{BT}(e^{j\omega}) = \sum_{k=-M}^{M} \hat{r}_x(k)w(k)e^{-j\omega k}$$

where $M < N$.

$$P_{BT}(e^{j\omega}) = \frac{1}{2\pi}\hat{P}_{per}(e^{j\omega}) * W(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{P}_{per}(e^{ju})W(e^{j(\omega-u)})du$$

$$E\left\{P_{BT}(e^{j\omega})\right\} = \frac{1}{2\pi}E\left\{\hat{P}_{per}(e^{j\omega})\right\} * W(e^{j\omega}) = \frac{1}{2\pi}P_x(e^{j\omega}) * W_B(e^{j\omega}) * W(e^{j\omega})$$

$$E\left\{P_{BT}(e^{j\omega})\right\} = \frac{1}{2\pi}P_x(e^{j\omega}) * W_{BT}(e^{j\omega})$$

If $M << N$ then $w_b(k)w(k) \approx w(k)$ and

$$E\left\{P_{BT}(e^{j\omega})\right\} \approx \frac{1}{2\pi}P_x(e^{j\omega}) * W(e^{j\omega})$$

$$Var\left\{\hat{P}_{BT}(e^{j\omega})\right\} \approx P_x^2(e^{j\omega})\frac{1}{N}\sum_{k=-M}^{M} w^2(k)$$

Performance Measures

	Variability $\nu = \frac{Var\left\{\hat{P}_x(e^{j\omega})\right\}}{E^2\left\{\hat{P}_x(e^{j\omega})\right\}}$	Resolution $\Delta\omega$	Figure of Merit $\mu = \nu\Delta\omega$
Periodogram	1	$0.89\frac{2\pi}{N}$	$0.89\frac{2\pi}{N}$
Bartlett	$\frac{1}{K}$	$0.89\frac{2\pi}{N}K$	$0.89\frac{2\pi}{N}$
Welch (%50)	$\frac{9}{8}\frac{1}{K}$	$1.28\frac{2\pi}{L}$	$0.72\frac{2\pi}{N}$
Blackman-Tukey	$\frac{2}{3}\frac{M}{N}$	$\frac{2\pi}{M}$	$0.43\frac{2\pi}{N}$

Resolution of Periodogram for $N = 20$ and $N = 64$

```
A=5; w1=0.4*pi; w2=0.45*pi; K=50;

N=[20]; n=transpose([0:N-1]); W = transpose(linspace(0,1,2*N)); Pav = zeros(2*N,1);
for i=1:K,
v=randn(N,1); v = ( v-mean(v) )/std(v);
x=A*sin(n*w1 ) + A*sin(n*w2) + v;
rx=conv(x,flipud(x))/(2*N+1);
j=sqrt(-1);   ew =  (-N+1:N-1);  Ew = transpose(linspace(0,1,2*N))*ew;  Ew=exp(-j*Ew*pi);
P= real(Ew*rx);
subplot(2,2,1);title('Periodogram Traces for N=20');hold;plot(W(2:end),10*log10(P(2:end)));grid;hold;
Pav = Pav + 10*log10(P);
end;
Pav=Pav/K;
subplot(2,2,2);title('Average of Periodogram Traces for N=20');hold;plot(W(2:end),Pav(2:end));

N=[64];  n=transpose([0:N-1]); W = linspace(0,1,2*N)'; Pav = zeros(2*N,1);
for i=1:K,
v=randn(N,1); v = ( v-mean(v) )/std(v);
x=A*sin(n*w1 ) + A*sin(n*w2) + v;
rx=conv(x,flipud(x))/(2*N+1);
j=sqrt(-1); ew =  (-N+1:N-1);  Ew = linspace(0,1,2*N)'*ew;  Ew=exp(-j*Ew*pi);
P= real(Ew*rx);
subplot(2,2,3);title('Periodogram Traces for N=64');hold;plot(W(2:end),10*log10(P(2:end)));hold;
Pav = Pav + 10*log10(P);
end;
Pav=Pav/K;
subplot(2,2,4);title('Average of Periodogram Traces for N=64');hold;plot(W(2:end),Pav(2:end));
```

Multitaper Spectral Analysis

$$\hat{P}_k(e^{j\omega}) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)w_k(n)e^{-j\omega n}\right|^2$$

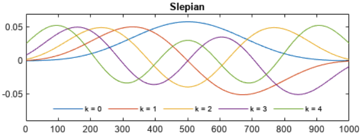
where $w_k(n)$ is Discrete Prolate Spheroidal (Slepian) Sequences (DPSS).

$$\hat{P}(e^{j\omega}) = \frac{1}{K}\sum_{k=1}^{K}\hat{P}_k(e^{j\omega})$$

DPSS Function

$$\sum_{m=0}^{N-1}\frac{\sin(\omega_c(n-m))}{\pi(n-m)}w_k(m) = \lambda_k w_k(n), k = 0, 1, \ldots, N-1$$

ω_c is the cut-off frequency for the band $[-\omega_c, \omega_c]$ in which the sequence $w_i(n)$ has maximal concentration of its energy.



Parameters for Generating DPSS Functions

N : The number of samples in the sequence

T : Duration of the sequence in sec

TB : Time-half-bandwidth product

The width of the main lobe $\Delta f = \frac{2TB}{T}$ determines the spectral resolution in Hz or $\Delta f = \frac{TB}{N}$ in per sample

K : Number of tapers which can be determined as $K = floor[2TB] - 1$

As the time-half-bandwidth increases or the length decreases, the smoothness increases and the resolution decreases.

MATLAB Function : `dpsf = dpss(N,TB,K);`

Comparison of Multitaper and Periodogram Spectra

```
n = [0:319]'; N=size(n,1);K=4;TB = 2;
x = cos(pi/4*n)+randn(size(n));
TP= dpss(N,TB,K);
X= repmat(x,1,K);
PS  = (abs(fft(X.*TP))).^2;
TP1= mean(PS,2);
TP1 =TP1([1:N/2+1]) ;
w1=linspace(0,1,size(TP1,1));
plot(w1,(TP1))
hold
PS0= (1/N)*(abs(fft(x))).^2;
TP0 =PS0([1:N/2+1]) ;
w1=linspace(0,1,size(TP0,1));
plot(w1,(TP0(1:N/2+1)))
grid
```

Effect of Time-half Bandwidth and number of Tapers on spectra

```
n = [0:319]';
N=size(n,1);
x = cos(pi/4*n)+randn(size(n));
K=4;TB=2;
TP= dpss(N,TB,K);
X= repmat(x,1,K);
PS  = (abs(fft(X.*TP))).^2;
TP1= mean(PS,2);
TS(:,1) =TP1([1:N/2]) ;
K=10; TB=2;
TP= dpss(N,TB,K);
X= repmat(x,1,K);
PS  = (abs(fft(X.*TP))).^2;
TP1= mean(PS,2);
TS(:,2) =TP1([1:N/2]) ;
K=10; TB=10;;
TP= dpss(N,TB,K);
X= repmat(x,1,K);
PS  = (abs(fft(X.*TP))).^2;
TP1= mean(PS,2);
TS(:,3) =TP1([1:N/2]) ;
w1=linspace(0,1,size(TS,1));
plot(w1,(TS(:,1:3)))
```

A signal sampled at 200Hz is analyzed with different tapers at different frequency bands such that

Band	Resolution	no. of tapers
$0-25Hz$	$2.5Hz$	1
$25-50Hz$	f+%10	2-3
$50-95Hz$	$5Hz$	3

```
SR = 200; %Sample Rate
TW = 0.4; % Duration of time window in sec
T = 2 ; % Total duration of signal in sec
N = T*SR; % total number of samples
x=filter([ 0.8 1 0.8  ],[1 -0.5  0.2],randn(N,1)); % data to be processed
delta_f = linspace(0,SR/2,1/TW);
t=(0:N-1)/SR; % Time index in sec
D = 0.05*SR; % sliding window overlap size in samples
L = TW * SR; % Number of samples in each window
MW =(N-L)/D ; % number of sliding windows
for i=0:MW, X(:,i+1) =x([1:L]+(i)*D); end;
Delta_f= [  ones(size([0:2.5:25]))*2.5  [25:25/18:50]*0.1 ones(size([50:5:100-5]))*5 ];
Delta_F= [   ( ([0:2.5:25])) [25:25/18:50]  ( ([50:5:100-5]))];
%L1=round(2*0.4*Delta_f)-1; %
L1=floor(2*0.4*Delta_f)-1; % Time Bandwidth
FF = [Delta_f' Delta_F' L1' (1:40)'] ;
% FF =  [Frequency increments    Frequency values    No. of Tapers  Integer Index ]
```

```
P =[1  find(abs( (diff(L1))))+1 ]; % index for taper number
%Single TAPER for [0-25] Hz with delta_f = 2.5 Hz
TP=repmat(dpss(L,L1(P(1))/2),1,size(X,2));
PS1 = abs(fft(X.*TP));
subplot(2,2,1);imagesc( (PS1(1:40,:))) ;axis xy ;  title('Single taper with 2.5 Hz Res.')
% TWO TAPERS for [25-50] Hz with delta_f =  0.1*Frequency
TP= (dpss(L,L1(P(2))/2));
PS2 = abs(fft(X.*repmat(TP(:,1),1,size(X,2)) + X.*repmat(TP(:,2),1,size(X,2))));
subplot(2,2,2);imagesc( (PS2(1:40,:))); axis xy ;   title('2-3 tapers with 2.5-5 Hz Res.')
% 3 TAPERs for [50-95] Hz with delta_f = 5 Hz
TP= (dpss(L,P(3)/2));
PS3 = abs(fft(X.*repmat(TP(:,1),1,size(X,2)) + X.*repmat(TP(:,2),1,size(X,2)) + X.*repmat(TP(:,3),1,size
subplot(2,2,3);imagesc( ((PS3(1:40,:)))); axis xy ;   title('3 tapers with 5 Hz Res.')
% Combined spectrogram with different resolutions in different bands
PS = [PS1(P(1):P(2)-1,:);PS2(P(2):P(3)-1,:) ;
PS3(P(3):size(Delta_f,2),:)];
subplot(2,2,4);imagesc( (PS)) ;  axis xy ;  title('Combination of spectra [0-25] [25-50] [50 95] ')
```

Minimum Variance Spectrum Estimation

$g_i(n)$ is an ideal band pass filter

$$\left|G_i(e^{j\omega})\right| = \begin{cases} 1 & ; \quad |\omega - \omega_i| < \Delta/2 \\ 0 & ; \quad otherwise \end{cases}$$

The filter output

$$P_i(e^{j\omega}) = P_x(e^{j\omega}) \left|G_i(e^{j\omega})\right|^2$$

The power

$$E\left\{|y_i(n)|^2\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_i(e^{j\omega})d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(e^{j\omega})\left|G_i(e^{j\omega})\right|^2 d\omega$$

$$E\left\{|y_i(n)|^2\right\} = \frac{1}{2\pi}\int_{\omega_i-\Delta/2}^{\omega_i+\Delta/2} P_i(e^{j\omega})d\omega \approx P_x(e^{j\omega_i})\frac{\Delta}{2\pi}$$

Power Spectral Density at ω_i is $\hat{P}_x(e^{j\omega}) = \frac{E\left\{|y_i(n)|^2\right\}}{\Delta/2\pi}$

$$G_i(e^{j\omega_i}) = \sum_{n=0}^{p} g_i(n) e^{-j\omega_i n} = 1,$$

$$\mathbf{g}_i = [g_i(0), g_i(1), \ldots g_i(p)]^H \ \mathbf{e}_i = [1, e^{j\omega_i}, \ldots e^{j\omega_i p}]^H$$

$$\min \left\{ E\left\{|y_i(n)|^2\right\} = \mathbf{g}_i^H \mathbf{R}_x \mathbf{g}_i \right\} \text{ with constraint } \mathbf{g}_i^H \mathbf{e}_i = 1$$

$$\mathbf{g_i} = \frac{\mathbf{R}_x^{-1}\mathbf{e}_i}{\mathbf{e}_i^H \mathbf{R}_x^{-1}\mathbf{e}_i} \text{ and } \min_{\mathbf{g}_i} E\left\{|y_i(n)|^2\right\} = \frac{1}{\mathbf{e}_i^H \mathbf{R}_x^{-1}\mathbf{e}_i}$$

$$\mathbf{g} = \frac{\mathbf{R}_x^{-1}\mathbf{e}}{\mathbf{e}^H \mathbf{R}_x^{-1}\mathbf{e}}$$

$$\sigma_x(\omega) = E\left\{|y_i(n)|^2\right\} = \mathbf{g}^H \mathbf{R}_x \mathbf{g} = \frac{1}{\mathbf{e}^H \mathbf{R}_x^{-1}\mathbf{e}}$$

Minimum variance spectrum estimation of white noise with $\mathbf{R}_x = \sigma_x^2 \mathbf{I}$

$\mathbf{g} = \frac{\mathbf{R}_x^{-1}\mathbf{e}}{\mathbf{e}^H \mathbf{R}_x^{-1}\mathbf{e}} = \frac{1}{p+1}\mathbf{e}$ $\qquad$ $\hat{\sigma}_x^2(\omega) = \frac{1}{\mathbf{e}^H \mathbf{R}_x^{-1}\mathbf{e}} = \frac{1}{p+1}\sigma_x^2$

$$\hat{P}_x(e^{j\omega}) = \frac{E\left\{|y_i(n)|^2\right\}}{\Delta/2\pi} = \frac{\sigma_x^2}{p+1}\frac{2\pi}{\Delta}$$

Setting $\Delta = \frac{2\pi}{p+1}$ we get $\hat{P}_{MV}(e^{j\omega}) = \frac{p+1}{\mathbf{e}^H \mathbf{R}_x^{-1}\mathbf{e}}$

MV estimate of an AR(1) process: $x(n) = \alpha x(n-1) + w(n)$; $|\alpha| < 1$ and $\sigma_w^2 = 1$

$r_x(k) = \frac{1}{1-\alpha^2}\alpha^{|k|}$ $\qquad P_x(e^{j\omega}) = \frac{1}{1+\alpha^2-2\alpha\cos\omega}$

p^{th} order minimum variance spectrum estimate is

$$\hat{P}_{MV}(e^{j\omega}) = \frac{p+1}{\mathbf{e}^H \mathbf{R}_x^{-1}\mathbf{e}}$$

$\mathbf{R}_x = \frac{1}{1-\alpha^2}\, Toep\,\{1, \alpha, \ldots, \alpha^p\}$

$$\mathbf{R}_x^{-1} = \begin{bmatrix} 1 & -\alpha & 0 & \cdots & 0 & 0 \\ -\alpha & 1+\alpha^2 & -\alpha & \cdots & 0 & 0 \\ 0 & -\alpha & 1+\alpha^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\alpha^2 & -\alpha \\ 0 & 0 & \cdots & 1+\alpha^2 & -\alpha & 1 \end{bmatrix}$$

$\mathbf{e}^h \mathbf{R}_x^{-1}\mathbf{e} = 2 + e^{j\omega}(p-2)(1+\alpha^2)e^{-j\omega} + -p\alpha(e^{j\omega} + e^{-j\omega})$

$$\hat{P}_{MV} = \frac{p+1}{2 + (p-1)/1 + \alpha^2 - 2\alpha p \cos\omega} \longrightarrow_{p\to\infty} P(e^{j\omega})$$

Maximum Entropy Method

Extrapolate the $r_x(k)$ for $|k| > p$ with the maximum entropy constraint *i.e.* $P_x(e^{j\omega})$ as flat as possible.

$$P_x(e^{j\omega}) = \sum_{k=-p}^{p} r_x(k)e^{-j\omega k} + \sum_{|k|>p} r_e(k)e^{-j\omega k}$$

Entropy

$$H(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln P_x(e^{j\omega}) d\omega$$

Constraint

$$P_x(e^{j\omega}) = \sum_{k=-p}^{p} r_x(k)e^{-j\omega k}, \quad |k| < p$$

$$\frac{\partial H(x)}{\partial r_e^*(k)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{P_x(e^{j\omega})} \frac{\partial P_x(e^{j\omega})}{\partial r_e^*(k)} d\omega = 0, \quad |k| > p$$

$$\frac{\partial P_x(e^{j\omega})}{\partial r_e^*(k)} = e^{j\omega k}, \ |k| > p$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{P_x(e^{j\omega})} e^{j\omega k} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} Q_x(e^{j\omega})e^{j\omega k} d\omega = 0 \rightarrow q_x(k) = 0 \text{ for } |k| > p$$

$$Q_x(e^{j\omega}) = \sum_{k=-p}^{p} q_x(k)e^{-j\omega k}$$

$$\hat{P}_{mem}(e^{j\omega}) = \frac{1}{\sum_{k=-p}^{p} q_x(k)e^{-j\omega k}}$$

Using spectral factorization

$$\hat{P}_{mem}(e^{j\omega}) = \frac{|b(0)|^2}{A_p(e^{j\omega})A_p^*(e^{j\omega})} = \frac{|b(0)|^2}{\sum_{k=1}^{p} |1 + a_p(k)e^{-j\omega k}|^2} = \frac{|b(0)|^2}{|\mathbf{e}^H \mathbf{a}_p|^2}$$

Normal Equations for $\mathbf{a}_p$

$$\begin{bmatrix} r_x(0) & r_x^*(1) & \cdots & r_x^*(p) \\ r_x(1) & r_x^*(0) & \cdots & r_x^*(p-1) \\ \vdots & \vdots & \vdots & \vdots \\ r_x(p) & r_x^*(p-1) & \cdots & r_x^*(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_p(0) \\ \cdots \\ a_p(p) \end{bmatrix} = \epsilon_p \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$|b(0)|^2 = r_x(0) + \sum_{k=1}^{p} a_p(k) r_x^*(k) = \epsilon_p$$

MEM estimate of $x(n) = A_1 e^{j\omega n_1} + w(n) \quad A_1 = |A_1| e^{j\phi}$ and $p(\phi) = 1/2\pi$ in $[-\pi, \pi]$

$$\mathbf{R}_x = P_1 \mathbf{e}_1 \mathbf{e}_1^H + \sigma_w^2 \mathbf{I} \text{ where } P_1 = |A_1|^2$$

$$R_x^{-1} = \frac{1}{\sigma_w^2}\left[I - \frac{P_1}{\sigma_w^2 + (p+1)P_1}\mathbf{e}_1\mathbf{e}_1^H\right] \qquad \mathbf{a}_p = \epsilon_p \mathbf{R}_x^{-1}\mathbf{u}_1 = \frac{\epsilon}{\sigma_w^2}\left[\mathbf{u}_1 - \frac{P_1}{\sigma_w^2 + (p+1)P_1}\mathbf{e}_1\right]$$

$$\hat{P}_{mem}(e^{j\omega}) = \frac{\epsilon_p}{|\mathbf{e}^H \mathbf{a}_p|^2} = \frac{\epsilon_p}{\left(\frac{\epsilon}{\sigma_w^2}\right)\left|\mathbf{e}^H\left(\mathbf{u}_1 - \frac{P_1}{\sigma_w^2 + (p+1)P_1}\mathbf{e}_1\right)\right|^2}$$

$$\hat{P}_{mem}(e^{j\omega}) = \frac{\epsilon_p}{|\mathbf{e}^H \mathbf{a}_p|^2} = \frac{\epsilon_p}{\left(\frac{\epsilon}{\sigma_w^2}\right)\left|\left(1 - \frac{P_1}{\sigma_w^2 + (p+1)P_1} W_R(e^{j(\omega-\omega_1)})\right)\right|^2}$$

$$a_p(0) = 1 = \left(\frac{\epsilon_p}{\sigma_w^2}\right)\left(1 - \frac{P_1}{\sigma_w^2 + (p+1)P_1}\right) \longrightarrow \epsilon_p = \sigma_w^2\left[1 + \frac{P_1}{\sigma_w^2 + pP_1}\right]$$

$$\hat{P}_{mem}(e^{j\omega}) = \frac{\sigma_w^2\left[1 - \frac{P_1}{\sigma_w^2 + (p+1)P_1}\right]}{\left|\left(1 - \frac{P_1}{\sigma_w^2 + (p+1)P_1} W_R(e^{j(\omega-\omega_1)})\right)\right|^2}$$

$$\max\left\{\hat{P}_{mem}(e^{j\omega})\right\} = \hat{P}_{mem}(e^{j\omega_1}) \approx p^2 \frac{P_1^2}{\sigma_w^2} \quad \text{if } P_1 >> \sigma_w^2$$

Parametric Power Spectrum Estimation Methods

Autoregressive Power Spectrum Estimation

$$\hat{P}_{AR}(e^{j\omega}) = \frac{|\hat{b}(0)|^2}{\left|1 + \sum_{k=1}^{p} \hat{a}_p(k) e^{j\omega k}\right|^2}$$

Autoregressive Moving Average Power Spectrum Estimation

$$\hat{P}_{ARMA}(e^{j\omega}) = \frac{\left|\sum_{k=0}^{q} \hat{b}(k) e^{-j\omega k}\right|^2}{\left|1 + \sum_{k=1}^{p} \hat{a}_p(k) e^{-j\omega k}\right|^2}$$

Autocorrelation Method (Yule-Walker Method)

$$\begin{bmatrix} r_x(0) & r_x^*(1) & \cdots & r_x^*(p) \\ r_x(1) & r_x^*(0) & \cdots & r_x^*(p-1) \\ \vdots & \vdots & \vdots & \vdots \\ r_x(p) & r_x^*(p-1) & \cdots & r_x^*(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_p(0) \\ \vdots \\ a_p(p) \end{bmatrix} = \epsilon_p \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$r_x(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x(n+k)x^*(n), \quad 0, 1, \ldots, p,$$

$$\epsilon_p = |b(0)|^2 = r_x(0) + \sum_{k=1}^{p} a_p(k) r_x^*(k)$$

Yule-Walker Method assumes that $x(n)$ is an AR process.
Maximum Entropy Method assumes that $x(n)$ is Gaussian.

Covariance Method

Forward prediction error $e_f(n) = x(n) + \sum_{k=1}^{p} a_p(k)x(n-k)$ is statistically minimized for $p \leq n \leq N$.

$$\begin{bmatrix} r_x(1,1) & r_x(2,1) & \cdots & r_x(p,1) \\ r_x(1,2) & r_x(2,2) & \cdots & r_x(p,2) \\ \vdots & \vdots & \vdots & \vdots \\ r_x(1,p) & r_x(2,p) & \cdots & r_x(p,p) \end{bmatrix} \begin{bmatrix} a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix} = - \begin{bmatrix} r_x(0,1) \\ r_x(0,2) \\ \vdots \\ r_x(0,p) \end{bmatrix}$$

$$r_x(k,l) = \sum_{n=p}^{N-1} x(n-l)x^*(n-k)$$

For short data records, covariance method yields higher resolution than autocorrelation method especially when $N >> p$.

Modified Covariance Method

Forward and backward $e_b(n) = x(n-p) + \sum_{k=1}^{p} a_p(k)x(N-p+k)$ prediction errors is minimized.

$$r_x(k,l) = \sum_{n=p}^{N-1} [x(n-l)x^*(n-k) + x(n-p+l)x^*(n-p+k)]$$

Modified Covariance Method yields statistically stable spectrum estimates with high resolution.

Model Order Selection Criteria

$$
\begin{aligned}
AIC(p) &= N \log \epsilon_p + 2p \\
MDL(p) &= N \log \epsilon_p + p \log N \\
FPE(p) &= \epsilon_p \frac{N+p+1}{N-p-1} \\
CAT(p) &= \left[\frac{1}{N} \sum_{j=1}^{p} \frac{N-j}{N\epsilon_j} \right] - \frac{N-p}{N\epsilon_p}
\end{aligned}
$$

$$e(n) = x(n) + \sum_{k=1}^{p} a_p(k)x(n-k); \quad n > 0$$

$$\mathbf{X}_p = \left[\begin{array}{ccccc} x(0) & 0 & 0 & \cdots & 0 \\ x(1) & x(0) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline x(p-1) & x(p-2) & x(p-3) & \cdots & x(0) \\ x(p) & x(p-1) & x(p-2) & \cdots & x(1) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x(N-1) & x(N-2) & x(N-3) & \cdots & x(N-p) \\ \hline x(N) & x(N-1) & x(N-2) & \cdots & x(N-p+1) \\ 0 & x(N) & x(N-1) & \cdots & x(N-p+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x(N) \end{array}\right] \longrightarrow \mathbf{Y}_p$$

$\mathbf{x}_a = [x(1), x(2), \ldots, x(N), 0, \ldots, 0]^T \quad \mathbf{a}_p = [a_p(1), \ldots, a_p(p)]^T$

$$\text{Autocorrelation Method}: \ \mathbf{e}_a = \mathbf{x}_a + \mathbf{X}_p\mathbf{a_p} \xrightarrow{\min\{\mathbf{e}_a^H\mathbf{e}_a\}} \underbrace{(\mathbf{X}_p^H\mathbf{X}_p)}_{\hat{\mathbf{R}}_x}\mathbf{a}_p = -\underbrace{\mathbf{X}_p^H\mathbf{x}_a}_{\hat{\mathbf{r}}_x}$$

$\mathbf{x}_c = [x(p), x(p+1), \ldots, x(N), 0, \ldots, 0]^T \quad \mathbf{a}_p = [a_p(1), \ldots, a_p(p)]^T$

$$\text{Covariance Method}: \ \mathbf{e}_c = \mathbf{x}_c + \mathbf{Y}_p\mathbf{a_p} \xrightarrow{\min\{\mathbf{e}_c^H\mathbf{e}_c\}} \underbrace{(\mathbf{Y}_p^H\mathbf{Y}_p)}_{\hat{\mathbf{R}}_x} = -\underbrace{\mathbf{Y}_p^H\mathbf{x}_c}_{\hat{\mathbf{r}}_x}$$

AR(4) Process: $x(n) = 2.7377x(n-1) - 3.7476x(n-2) + 2.6293x(n-3) - 0.9224x(n-4) + w(n)$

Spectrum Estimation Using Autocorrelation and Covariance Method

```
N=128; K=50; p=4;  a = [2.7377 -3.7476  2.6293 -0.9224 ];
% Generate AR(4) process
w=randn(N,K);w=(w-repmat(mean(w),N,1))./std(w); for i=1:K,x(:,i) = filter(1,[1 -a],w(:,i));end;
% Generate Frequency Vector for Fourier Transform
j=sqrt(-1); Nw=100; ew = transpose(linspace(0,1,Nw));  Ew = ew*[0:p];  Ew=exp(-j*Ew*pi);
P_av=zeros(size(Ew,1),1); P_avc=zeros(size(Ew,1),1);
for i=1:K
% Autocorrelation Method  and Power Spectrum  Estimation
X= convmtx(x(:,i),p); x_p = [ x(2:end,i) ;zeros(p,1) ]; a_p = -X\x_p; e = std( x_p + X*a_p);
P=Ew*[1; a_p]; P = ones(size(P))./abs(P); P=(e*P).^2;
subplot(2,2,1);title('Autocorrelation Method') ;hold;plot(ew,10*log10(P)); hold;
P_av = P_av + 10*log10(P);
%  Covariance  Method  and Power Spectrum  Estimation
Yp = X([p:N-1],:); yp = x_p([p:N-1]); a_p = -Yp\yp; e = std( yp + Yp*a_p);
P=Ew*[1; a_p]; P = ones(size(P))./abs(P); P=(e*P).^2;
subplot(2,2,3);title('Covariance Method') ;hold;plot(ew,10*log10(P));hold;
P_avc = P_avc + 10*log10(P);
end;
subplot(2,2,1);grid;axis([0 1 -20 50]);subplot(2,2,3);grid;axis([0 1 -20 50]);
P_av=P_av/K; P_avc=P_avc/K;
% Power Spectrum Using real AR(4)  parameters
P_a=Ew*[1; -transpose(a)]; P_a = ones(size(P_a))./abs(P_a); P=(1*P_a);
subplot(2,2,2);title('Real and Aurocorrelation Estimated PSD') ;hold;
plot(ew,[  20*log10(P_a)  (P_av) ]);grid;axis([0 1 -20 50])
subplot(2,2,4);title('Real and Covariance Estimated PSD') ;hold;
plot(ew,[  20*log10(P_a)  (P_avc) ]);grid;axis([0 1 -20 50]);
```

Eigendecomposition Based Frequency Estimation

$x(n) = A_1 e^{j\omega_1 n} + w(n)$ where $A_1 = |A_1| e^{j\phi}$ and $p(\phi) = \frac{1}{2\pi}$ over $[-\pi, \pi]$

$$r_x(k) = P_1 e^{j\omega_1 k} + \sigma_w^2 \delta(k), \;\; P_1 = |A_1|^2$$

$$\mathbf{R}_x = \mathbf{R}_s + \mathbf{R}_w$$

$\mathbf{R}_s = P_1 \, Toep\{1, e^{j\omega_1}, e^{j2\omega_1}, \ldots, e^{j(M-1)\omega_1}\}$ and $\mathbf{R}_w = \sigma_\omega^2 \mathbf{I}$
$\mathbf{e}_1 = [1, e^{j\omega_1}, e^{j2\omega_1}, \ldots, e^{j(M-1)\omega_1}]^T$
$\mathbf{R}_s = P_1 \mathbf{e}_1 \mathbf{e}_1^H$ and $rank(\mathbf{R}_s) = 1.$
$\mathbf{R}_s \mathbf{e}_1 = M P_1 \mathbf{e}_1$
$\lambda_i(\mathbf{R}_s) = \{P_1, 0, \ldots, 0\}$, $\mathbf{v}_i = \{\mathbf{e}_1, \mathbf{v}_2, \ldots, \mathbf{v}_M\}$ and $\mathbf{e}_1^H \mathbf{v}_i = 0$
$\lambda_i(\mathbf{R}_x) = \{P_1 + \sigma_w^2, \sigma_w^2, \ldots, \sigma_w^2\}$, $\mathbf{v}_i = \{\mathbf{e}_1, \mathbf{v}_2, \ldots, \mathbf{v}_M\}$ and $\mathbf{e}_1^H \mathbf{v}_i = 0$

1. Peform eigendecomposition of $\mathbf{R}_x$.
2. Assign $\sigma_w^2 = \lambda_{min}$ and $P_1 = (\lambda_{max} - \lambda_{min})/M$
3. Determine ω_1 using $\omega_1 = \angle \mathbf{v}_{max}(1)$

$\mathbf{e}^H\mathbf{v}_i = V_i(e^{j\omega}) = \sum_{k=0}^{M-1} v_i(k)e^{-jk\omega}$ and $V_i(e^{j\omega_1}) = 0$ because $\mathbf{e}_1^H\mathbf{v}_i = 0$

A frequency estimation function can be defined as

$$\hat{P}_i(e^{j\omega}) = \frac{1}{\left|\sum_{k=0}^{M-1} v_i(k)e^{-j\omega k}\right|^2} = \frac{1}{|\mathbf{e}^H\mathbf{v}_i|^2}$$

To overcome the sensitivity error to noisy estimate of $r_x(k)$ averaging is performed.

$$\hat{P}(e^{j\omega}) = \frac{1}{\sum_{i=2}^{M} \alpha_i \left|\mathbf{e}^H\mathbf{v}_i\right|^2}$$

For $x(n) = A_1 e^{j\omega_1 n} + A_2 e^{j\omega_2 n} + w(n)$

$r_x(k) = P_1 e^{j\omega_1 k} + P_2 e^{j\omega_2 k} + \sigma_w^2 \delta(k)$ and $\mathbf{R_x} = P_1\mathbf{e}_1\mathbf{e}_1^H + P_2\mathbf{e}_2\mathbf{e}_2^H + \sigma_w^2\mathbf{I}$

$\mathbf{R_x} = \mathbf{EPE}^H + \sigma_w^2\mathbf{I}$ where $\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2]$ and $\mathbf{P} = diag\{P_1, P_2\}$

$$\underbrace{\lambda_1 \geq \lambda_2}_{signal} \geq \underbrace{\lambda_3 \ldots, \geq \lambda_M}_{noise}$$

$\mathbf{e}_1^H\mathbf{v}_i = 0$; $i = 3, 4, \ldots, M$ and $\mathbf{e}_2^H\mathbf{v}_i = 0$; $i = 3, 4, \ldots, M$

$$\hat{P}(e^{j\omega}) = \frac{1}{\sum_{i=3}^{M} \alpha_i \left|\mathbf{e}^H\mathbf{v}_i\right|^2}$$

For p sinusiods in noise

$$\mathbf{R_x} = \sum_{i=1}^{p} (\lambda_i^s + \sigma_w^2)\mathbf{v}_i\mathbf{v}_i^H + \sum_{i=p+1}^{M} (\sigma_w^2)\mathbf{v}_i\mathbf{v}_i^H$$

$$\mathbf{R}_x = \mathbf{V}_{ss}\mathbf{V}_s^H + \mathbf{V}_{ww}\mathbf{V}_w^H$$

where $\mathbf{V}_s = [\mathbf{v}_1, \ldots, \mathbf{v}_p]$, $\mathbf{V}_w = [\mathbf{v}_{p+1}, \ldots, \mathbf{v}_M]$, $\mathbf{V}_{ss} = diag\{\lambda_i^s + \sigma_w^2\}$ and $\mathbf{V}_{ww} = diag\{\sigma_w^2\}$

$$\hat{P}(e^{j\omega}) = \frac{1}{\sum_{i=p+1}^{M} \alpha_i \left|\mathbf{e}^H\mathbf{v}_i\right|^2}$$

MUSIC Algorithm

$\hat{\sigma}_w^2 = \frac{1}{M-p} \sum_{k=p+1}^{M} \lambda_k$

$\mathbf{e}_i^H\mathbf{v}_j = 0$ for $i = 1, 2, \ldots, p$ and $j = p+1, p+2, \ldots, M-1$

$V_j(z) = \sum_{k=0}^{M-1} v_i(k) z^{-k}$ has p zeros corresponding to signal frequencies.

$$\hat{P}_{Music}(e^{j\omega}) = \frac{1}{\sum_{i=p+1}^{M} \left|\mathbf{e}^H\mathbf{v}_i\right|^2}$$

Amplitude Estimation in MUSIC Algorithm

$$\mathbf{v}_i^H \mathbf{R}_x \mathbf{v}_i = \mathbf{v}_i^H \left\{ \sum_{k=1}^{p} P_k \mathbf{e}_k \mathbf{e}_k^H + \sigma_w^2 \mathbf{I} \right\} \mathbf{v}_i = \lambda_i,\ i = 1, 2, \ldots, p$$

$$\mathbf{e}_k^H \mathbf{v}_i = V_i(e^{j\omega_k})$$

$$\sum_{k=1}^{p} P_k |V_i(e^{j\omega_k})|^2 = \lambda_i - \sigma_w^2, i = 1, 2, \ldots, p$$

$$\begin{bmatrix} |V_1(e^{j\omega_1})|^2 & |V_1(e^{j\omega_2})|^2 & \cdots & |V_1(e^{j\omega_p})|^2 \\ |V_2(e^{j\omega_1})|^2 & |V_2(e^{j\omega_2})|^2 & \cdots & |V_2(e^{j\omega_p})|^2 \\ \vdots & \vdots & \vdots & \vdots \\ |V_p(e^{j\omega_1})|^2 & |V_p(e^{j\omega_2})|^2 & \cdots & |V_p(e^{j\omega_p})|^2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_p \end{bmatrix} = \begin{bmatrix} \lambda_1 - \sigma_w^2 \\ \lambda_2 - \sigma_w^2 \\ \vdots \\ \lambda_p - \sigma_w^2 \end{bmatrix}$$

MUSIC Spectrum of a signal with two sinusoids in Noise

```
A=5; w1=0.4*pi; w2=0.45*pi; N=[64]; K=50;n=[0:N-1]'; M=19;
NW=128; W = linspace(0,1,NW)'; Pav = zeros(NW,1); j=sqrt(-1);
for i=1:K,
v=randn(N,1); v = ( v-mean(v) )/std(v);
x=A*sin(n*w1 ) + A*sin(n*w2) + v;
rx=conv(x,flipud(x))/N;
Rx = toeplitz(rx(N+[0:M]));
[V,Lambda] = eig(Rx);
Lambda = diag(Lambda);
ew =  [0:M];  Ew = linspace(0,1,NW)'*ew;  Ew=exp(-j*Ew*pi);
P= abs(Ew*V(:,1:end-4)).^2; P=sum(P,2); P=ones(size(P))./P;
subplot(2,2,1);hold;plot(W(1:end) ,10*log10(P(1:end)));hold;
Pav = Pav + 10*log10(P);
end;
grid;
Pav=Pav/K; subplot(2,2,2);plot(W(1:end),Pav(1:end));grid
```

Principal Components Spectrum Estimation

$$\mathbf{R}_x = \sum_{i=1}^{M} \lambda_i \mathbf{v}_i \mathbf{v}_i^H = \underbrace{\sum_{i=1}^{p} \lambda_i \mathbf{v}_i \mathbf{v}_i^H}_{\hat{\mathbf{R}}_s} + \sum_{i=p+1}^{M} \lambda_i \mathbf{v}_i \mathbf{v}_i^H$$

Blackman-Tukey Frequency Estimation Methods

$$\hat{P}_{BT}(e^{j\omega}) = \sum_{k=-M}^{M} \hat{r}_x(k) w(k) e^{-j\omega k} = \frac{1}{M} \sum_{k=-M}^{M} (M - |k|) \hat{r}_x(k) e^{-j\omega k} = \frac{1}{M} \mathbf{e}^H \mathbf{R}_x \mathbf{e}$$

$$= \frac{1}{M} \sum_{i=1}^{M} \lambda_i |\mathbf{e}^H \mathbf{v}_i|^2 \longrightarrow \hat{P}_{PC-BT}(e^{j\omega}) = \frac{1}{M} \sum_{i=1}^{p} \lambda_i |\mathbf{e}^H \mathbf{v}_i|^2$$

Minimum Variance Frequency Estimation

$$\hat{P}(e^{j\omega}) = \frac{M}{\mathbf{e}^H \mathbf{R}_x^{-1} \mathbf{e}}$$

$$\mathbf{R}_x^{-1} = \sum_{i=1}^{p} \frac{1}{\lambda_i} \mathbf{v}_i \mathbf{v}_i^H + \sum_{i=p+1}^{M} \frac{1}{\lambda_i} \mathbf{v}_i \mathbf{v}_i^H \longrightarrow \hat{P}_{PC-MV}(e^{j\omega}) = \frac{M}{\sum_{i=1}^{p} \frac{1}{\lambda_i} |\mathbf{e}^H \mathbf{v}_i|^2}$$

Autoregressive Frequency Estimation

Generic Equation to solve $\longrightarrow$ $\mathbf{R}_x \mathbf{a}_M = \epsilon_M \mathbf{u}_1$ and $\mathbf{R}_x$ is $M+1 \times M+1$

$$\mathbf{a}_M = \epsilon_M \mathbf{R}_x^{-1} \mathbf{u}_1$$

$$\hat{P}_{AR}(e^{j\omega}) = \frac{|b(0)|^2}{|\mathbf{e}^H \mathbf{a}_M|^2}$$

$$\mathbf{a}_{PC} = \epsilon_M \left(\sum_{i=1}^{p} \frac{1}{\lambda_i} \mathbf{v}_i \mathbf{v}_i^H \right) \mathbf{u}_1 = \epsilon_M \sum_{i=1}^{p} \frac{1}{\lambda_i} v_i(0)^* \mathbf{v}_i =$$

$$\hat{P}_{AR}(e^{j\omega}) = \frac{1}{\left| \sum_{i=1}^{p} \alpha_i \mathbf{e}^H \mathbf{v}_i \right|^2}$$