Multirate Signal Processing

Lecture Notes

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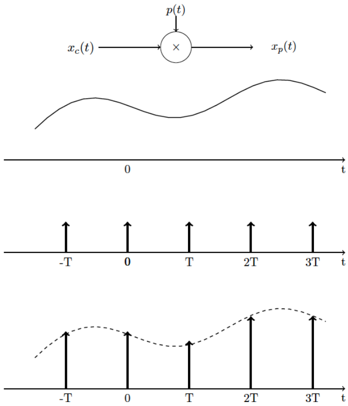
Some concepts and illustrations in this lecture are adapted from the textbooks,

Signals and Systems, 2nd Edition by Alan Oppenheim, Alan Willisky and H. Nawab, *Prentice Hall*.

Applied Digital Signal Processing, Dimitris G. Manolakis & K. Ingle, *Cambridge*.

Nyquist Sampling Theorem



$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

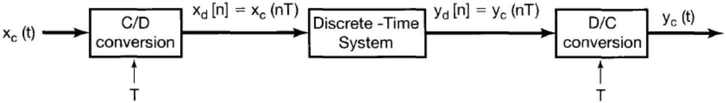
$$x_p(t) = x_c(t)p(t)$$

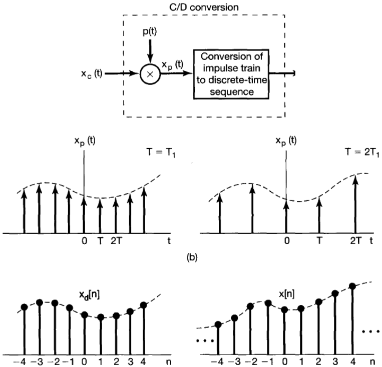
$$X_p(\Omega) = \frac{1}{2\pi} X_c(\Omega) * P(\Omega)$$

$$P(\Omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\Omega - n\frac{2\pi}{T})$$

$$X_p(\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(\Omega - n\frac{2\pi}{T})$$

Analog to Digital Conversion





$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) \longleftrightarrow X_p(\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega nT}$$

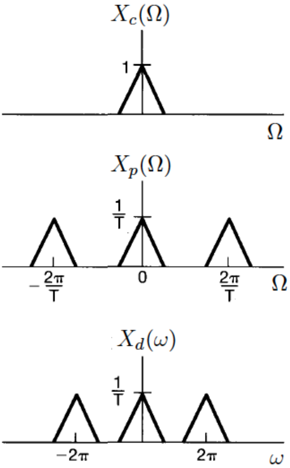
Discrete Fourier Transform of $X_d[n]$ is

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_c[nT]e^{-j\omega n}$$

Comparing $X_p(\Omega)$ and $X_d(\omega)$,

$$X_d(\omega) = X_p(\Omega)\Big|_{\Omega=\frac{\omega}{T}}$$

$$X_p(\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(\Omega - n\frac{2\pi}{T}) \text{ and } X_d(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(\omega - 2\pi n)$$



When $x_c(t)$ is converted into $x_d[n]$, the time axis is scaled by $1/T$ which leads to a scaling of frequency axis by T.

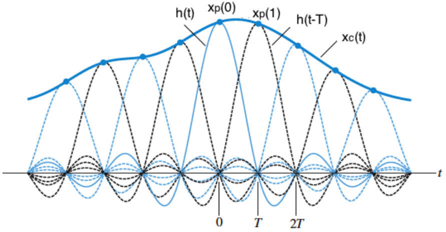
Reconstruction of $x_c(t)$ from $x_p(t)$

$x_c(t)$ can be recovered from $x_p(t)$ by a LP filter $H(\Omega)$ with cut-off frequency $\Omega_s = \frac{2\pi}{T}$ as $H(\Omega) = \begin{cases} T & -\Omega_s/2 \leq \Omega \leq \Omega_s/2 \\ 0 & otherwise \end{cases}$

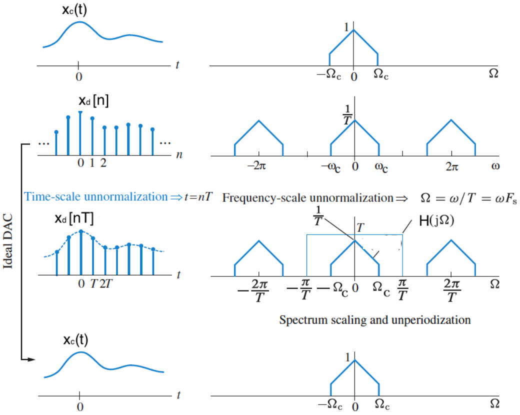
$h(t) = \frac{\sin(\pi t/T)}{\pi t/T} = sinc(\frac{t}{T})$

$x_c(t) = x_p(t) * h(t)$

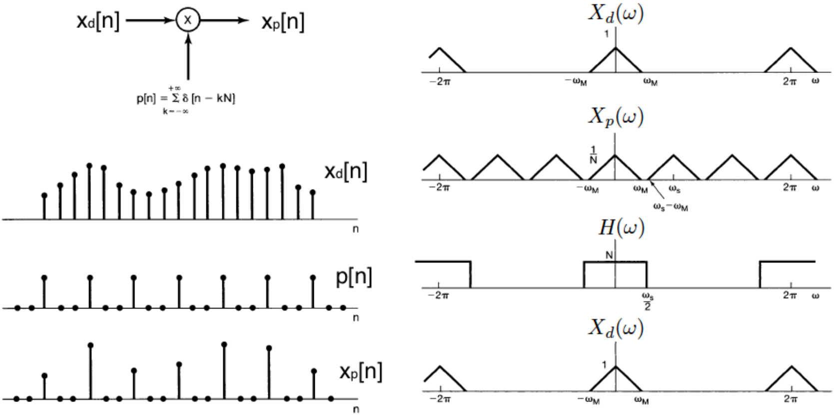
$$x_c(t) = \sum_{n=-\infty}^{\infty} x_p(nT)\delta(t-nT) * sinc(\frac{t}{T}) = \sum_{n=-\infty}^{\infty} x_p(nT) sinc(\frac{t-nT}{T})$$



Reconstruction of $x_c(t)$ from $x_d[n]$



Sampling and Reconstruction of Discrete Signals

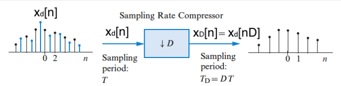


$$X_p(\omega) = \frac{1}{2\pi}\left[X_d(\omega) * P(\omega)\right] = \frac{1}{2\pi}\left[X_d(\omega) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\delta(\omega - \omega_s k)\right]$$

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty} X_d(\omega - \omega_s k) \text{ where } \omega_s = \frac{2\pi}{N}$$

$$X_d(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(\omega - 2\pi n) \text{ or } X_d(\Omega T) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c\left(\Omega - \frac{2\pi}{T} n\right)$$

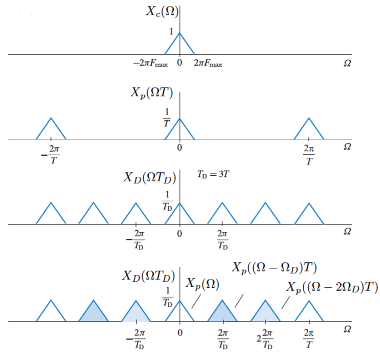
Relation between the spectra of $x_c(t)$ and $x_p[n]$



$$X_D(\Omega T_D) = \frac{1}{T_D} \sum_{n=-\infty}^{\infty} X_c(\Omega - n\Omega_D)$$

$$= \frac{1}{DT} \sum_{n=-\infty}^{\infty} X_c\left(\Omega - n\frac{2\pi}{DT}\right)$$

$$T_D = DT \text{ and } \Omega_D = \frac{2\pi}{T_D}$$



$$X_D(\Omega T_D) = \frac{1}{D} \sum_{m=0}^{D-1} \left[\frac{1}{T} \sum_{n=-\infty}^{\infty} X_c\left(\Omega - n\frac{2\pi}{T} - m\frac{2\pi}{DT}\right)\right] = \frac{1}{D} \sum_{m=0}^{D-1} X_p((\Omega - m\Omega_D)T)$$

Downsampling by a factor of 2

The Fourier transform of a discrete sequence $x_d[n]$ is

$$X_d[\omega] = \sum_{n=-\infty}^{\infty} x_d[n] e^{-j\omega n}$$

Downsampling $x_d[n]$ yields a sequence $x_D[n] = \{\ldots, x_d[-2], x_d[0], x_d[2], \ldots\}$

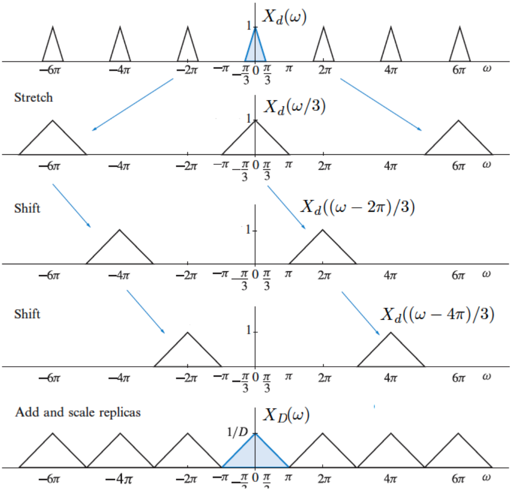
$X_D(\omega) = \cdots + x_d[-2]^{j\omega} + x_d[0] + x_d[2]e^{-j\omega} + \ldots \; = \sum_{k=-\infty}^{\infty} x_d[2k]e^{-j\omega k}$

If we define $x_u[n] = \frac{1}{2} x_d[k] + (-1)^k x_d[k]$

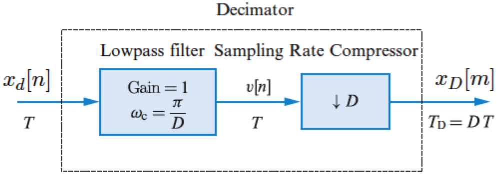
$X_D(\omega) = \sum_{k=-\infty}^{\infty} x_u[2k]e^{-j\omega k} = X_u(\omega/2)$

$X_u(\omega) = \frac{1}{2} \sum_{k=-\infty}^{\infty} [x_d[k] + (-1)^k x_d[k]]e^{-j\omega k} = \frac{1}{2}[X_d(\omega) + X_d(\omega - \pi)]$

$X_D(\omega) = X_u(\omega/2) = \frac{1}{2}[X_d(\omega/2) + X_d(\omega/2 - \pi)]$



$$X_D(\omega) = \frac{1}{3}[X_p(\omega/3) + X_p(\omega/3 - 2\pi/3) + X_p(\omega/3 - 4\pi/3)]$$



$$x_D[m] = v[mD] = \sum_{k=0}^{M} h[k] x_d[mD - k]$$

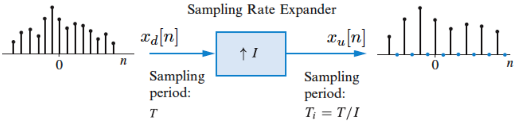
The continuous-time signal $x_c(t)$ can be perfectly reconstructed from the samples $x_D[m]$ using the bandlimited interpolation formula

$$x_c(t) = x_D[m] * sinc(t/TD) = \sum_{m=-\infty}^{\infty} x_D[m] \frac{\sin(\pi(t/TD - m))}{\pi(t/DT - m)}$$

Since $x_p[n] = x_c(t)|_{t=nT}$, perfect reconstruction of $x_p[n]$ is possible

$$x_d[n] = \sum_{m=-\infty}^{\infty} x_D[m] \frac{\sin(\pi(n/D - m))}{\pi(n/D - m)}$$

Upsampling



The upsampled signal $x_I[n] \triangleq x_c(nT_I) = x_c(nT/I)$

We can obtain $x_I[n] = x_d[m] * \frac{\sin(\pi(n/I))}{\pi(n/I)} = \sum_{m=-\infty}^{\infty} x_d[m] \frac{\sin(\pi(n-mI)/I)}{\pi(n-mI)/I}$

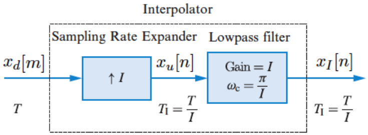
The interpolator function has the general property

$$x_I[n] = \sum_{m=-\infty}^{\infty} x_d[m]h[n-mI], \; h[n] = \begin{cases} 1 & n = 0 \\ 0 & n = \pm I, \pm 2I, \ldots \end{cases}$$

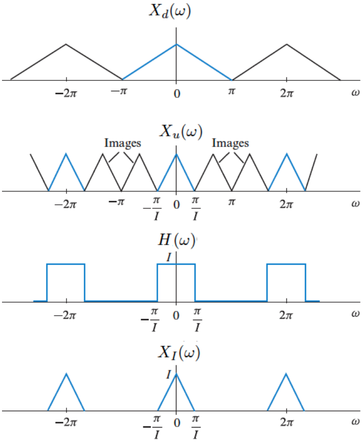
$$X_I(\omega) = \sum_{m=-\infty}^{\infty} x_d[m]H(\omega)e^{-jI\omega m} = H(\omega) \sum_{m=-\infty}^{\infty} x_d[m]e^{-jI\omega m} = H(\omega)X(I\omega)$$

We define $x_u[n]$ by inserting $I-1$ zeros between samples of $x_d[n]$

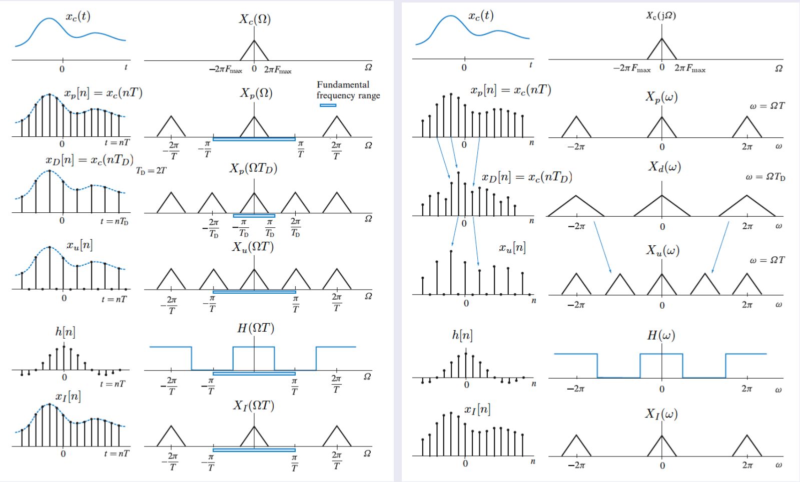
$$x_u[n] \triangleq \begin{cases} x_d[n/I] & n : \text{ multiple of } I \\ 0 & \text{otherwise} \end{cases} \longleftrightarrow X_u(\omega) = X(I\omega)$$



$$X_I(\omega) = H(\omega)X(I\omega)$$
$$x_I[n] = x_u[n] * h[n]$$



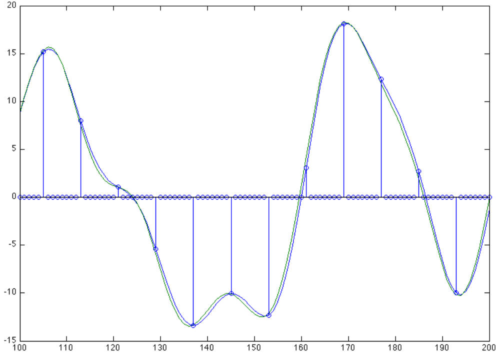
Decimation and interpolation operations in time and frequency domains



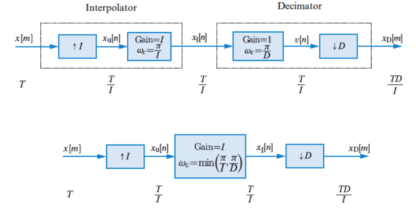
Tutorial for Interpolation of a downsampled signal using sinc function

```
% Generate a signal as a sum of sinusiods
Nf=5; % number of sinusoids
Fs = 400; % Original sampling rate
Td=1; % signal duration
Af = [10 6 5 4 3]' ; %  amplitudes of the sinusoids
f = [ 5  8 10 12 20]; % frequencies of individual components in the signal
T  =1/Fs; % sampling interval of the original signal
t = linspace(0,Td,Td*Fs)'; % time axis of the resampled signal
x =  sin(t*2*pi*f)*Af; % original signal sampled as a sum of the sinusoids
D = 8 ; % downsample by a factor of D
td=1:D:length(x); % downsampled time index
L = length(td) ; % number of samples in the downsampled signal
M = length(t) ;  % number of samples in the original signal
xd=x(td); % downsampled version of the original signal by a factor of D
% Convolution matrix of sinc interpolation function  sampled at every T secs
S = sinc( (t*ones(1,L)  - ones(M,1)*(td-1)*T)/(D*T) );
x_int = S*xd; %  interpolated version of the downsampled signal
x_ups = zeros(size(x)); x_ups(1:D:end) = xd; % upsampled version of the downsampled signal
t1=100:200; % extracting a poriton of the signal for plotting
plot(t1,x(t1),t1,x_int(t1)); hold ; stem(t1,x_ups(t1))
```

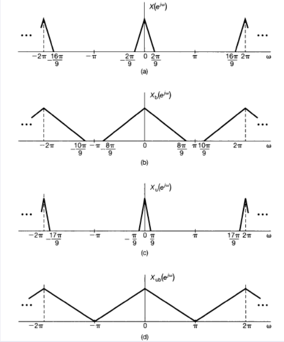
Fractional delay of a sample



Sampling rate change by a fractional factor



$$H(\omega) = \begin{cases} I & 0 \leq |\omega| \min(\frac{\pi}{I}, \frac{\pi}{D}) \\ 0 & \min(\frac{\pi}{I}, \frac{\pi}{D}) \leq |\omega| \leq \pi \end{cases}$$



Original spectrum

Spectrum after downsampling by 4

Spectrum after upsampling $x[n]$ by 2

Spectrum after upsampling $x[n]$ by 2

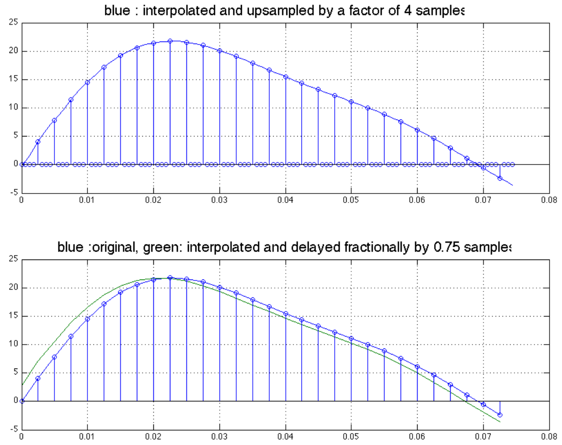
then downsampling by 9.

Fractional delay of a sample

```
N0 = 0.75 % amount of fractional shift of a sampling interval T
D =round(1/min([1-N0 N0])) ;
Fs1 = Fs*D; % Sampling rate after upsampling
T  =1/Fs1; % sampling interval of the original signal
Td=1; % signal duration
n =  [1:length(x)] ;  % sampled original  signal time index
t = linspace(0,Td,Td*Fs1)'; % time axis of the resampled signal
t_ups=1:D*length(x); % upsampled time axis
x_ups = zeros(size(t_ups)); x_ups(1:D:end) = x; % upsampled version of the original sampled signal
M = length(t_ups) ; % number of samples in the upsampled signal
L = length(x) ;  % number of samples in the original signal
% interpolation functions sampled at every T/D secs
S = sinc( (  t*ones(1,L)  - ones(M,1)*(n-1)*T*D   )  / (D*T) );  % sinc((t-nT)T)
x_int = S*x; %  interpolated version of the resampled signal
x_d  = x_int(1+(round(D*N0)):D:end);  % downsampled signal  with a fractional delay

t1=1:30; % extracting a time interval for the original signal
% extracting the same time interval for the upsampled and interpolated signal
t2=[(D*(min(t1)-1)+1 ):D*max(t1)];
t0=t1-1;
subplot(2,1,2);plot(t0/Fs,x(t1),t0/Fs,x_d(t1));hold ;stem(t0/Fs,x(t1)); grid ;hold
title([ 'blue :original, green: interpolated and delayed fractionally by ' num2str(N0) ' samples' ] );
t0=t2-1;
subplot(2,1,1);plot(t0/Fs1,x_int(t2)); hold ;stem(t0/Fs1,x_ups(t2));hold;grid ;
title([ 'blue : interpolated and upsampled by a factor of ' num2str(D) ' samples' ] )
```

Fractional delay of a sample



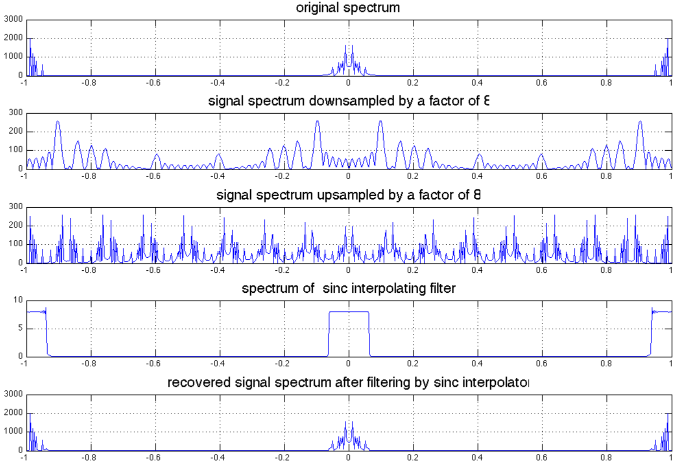
Frequency spectra of an original, downsampled and interpolated signals

```
Nf=5; % number of sinusoids
Fs = 400; % Original sampling rate
Td=1; % signal duration
Af = [10 6 5 4 3]' ; % amplitudes of the sinusoids
f = [ 5 8 10 12 20]; % frequencies of individual components in the signal
T =1/Fs; % sampling interval of the original signal
t = linspace(0,Td,Td*Fs)'; % time axis of the resampled signal
x = sin(t*2*pi*f)*Af; % original signal sampled as a sum of the sinusoids
D = 8 ; % downsample by a factor of D
td=1:D:length(x); % downsampled time index
L = length(td) ; % number of samples in the downsampled signal
M = length(t) ; % number of samples in the original signal
xd=x(td); % downsampled version of the original signal by a factor of D
% Convolution matrix of sinc interpolation function sampled at every T secs
S = sinc( (t*ones(1,L) - ones(M,1)*(td-1)*T)/(D*T) );
x_int = S*xd; % interpolated version of the downsampled signal
x_ups = zeros(size(x)); x_ups(1:D:end) = xd;
```

Frequency spectra of an original, downsampled and interpolated signals

```
Df = 800; % sampling rate of frequency spectrum
w = linspace(-1,1,Df);
% Estimating frequency responses
H_x = freqz(x,1,w*2*pi);
H_xups = freqz(x_ups,1,w*2*pi);
H_xd = freqz(xd,1,w*2*pi);
H_I = freqz(S(:,end/2),1,w*2*pi);
H_xint = freqz(x_int ,1,w*2*pi);
subplot(5,1,1); plot(w,abs(H_x));grid; title('original spectrum','fontsize',16)
subplot(5,1,2); plot(w,abs(H_xd));
grid;title('signal spectrum downsampled by a factor of 8','fontsize',16);
subplot(5,1,3); plot(w,abs(H_xups));
grid;title('signal spectrum upsampled by a factor of 8','fontsize',16);
subplot(5,1,4); plot(w,abs(H_I));
grid;title('spectrum of sinc interpolating filter','fontsize',16);
subplot(5,1,5); plot(w,abs(H_xint));
grid;title('recovered signal spectrum after filtering by sinc interpolator','fontsize',16);
```

Frequency spectra of an original, downsampled and interpolated signals



Sampling rate compressor

Sampling train : $\delta_D[n] = \sum_{k=-\infty}^{\infty} \delta[n-kD] = \frac{1}{D}\sum_{k=0}^{D-1} W_D^{-kn}$, $W_D = e^{-\frac{j2\pi}{D}}$

$z-$transform of sampled sequence $v[n] = x[n]\delta_D[n]$:

$$V(z) = \sum_{n=-\infty}^{\infty} x[n]\delta_D[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{D}\sum_{k=0}^{D-1} W_D^{-kn}\right]z^{-n}$$

$$= \frac{1}{D}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{D-1} x[n](W_D^k z)^{-n} = \frac{1}{D}\sum_{k=0}^{D-1} X(W_D^k z)$$

Upsampler output $y[n] = v[nD] = x[nD]$ and its $z-$transform

$$Y(z) = \sum_{n=-\infty}^{\infty} v[nD]z^{-n} = \sum_{m=-\infty}^{\infty} v[m]z^{-m/D} = V(z^{1/D}) = \frac{1}{D}\sum_{k=0}^{D-1} X(W_D^k z^{1/D})$$

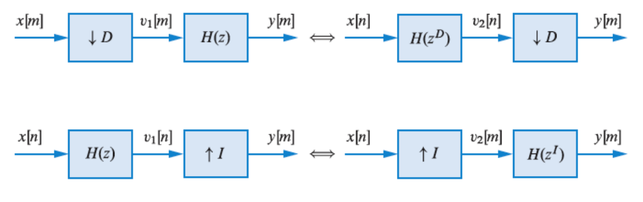
Sampling rate expander

$$y[n] = \begin{cases} x[n/I] & n = 0, \pm I, \pm 2I, \dots \\ 0 & otherwise \end{cases}$$

The $z-$transform of the output

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n/I]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-mI} = X(z^I)$$

Multirate identities: Interchangibility of up or down sampling with filtering



$Y(z) = H(z)V_1(z) = H(z)\frac{1}{D}\sum_{k=0}^{D-1} X(W_D^k z^{1/D}) \Leftarrow$

$V_2(z) = H(z^D)X(n)$ and $Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} V_2(W_D^k z^{1/D})$

$Y(z) = H(z)\frac{1}{D}\sum_{k=0}^{D-1} X(W_D^k z^{1/D})$ since $H(W_D^{kD} z^{D/D}) = H(z)$

$V_1(z) = H(z)X(z)$ and $Y(z) = V_1(z^I)$ and $Y(z) = H(z^I)X(z^I)$

$V_2(z) = X(z^I)$ and $Y(z) = H(z^I)V_2(z)$ and $Y(z) = H(z^I)X(z^I)$

Polyphase Filters

An FIR filter with $N = ML$ is realized with M filters each having lenght L as for instance

$N = 6$ and $M = 2$

$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5}$

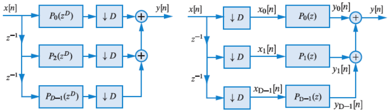
$= \left(h[0] + h[2]z^{-2} + h[4]z^{-4}\right) + z^{-1}\left(h[1] + h[3]z^{-2} + h[5]z^{-4}\right)$

$$H(z) = P_0(z^2) + z^{-1}P_1(z^2),$$

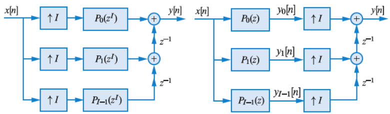
In general,

$p_k[n] \triangleq h[nM + k],\ k = 0, 1, 2, \ldots, M - 1$

$H(z) = \sum_{k=0}^{M-1} z^{-k} P_k[z^M],\ P_k(z) = \sum_{n=0}^{L-1} P_k[n]z^{-n}$



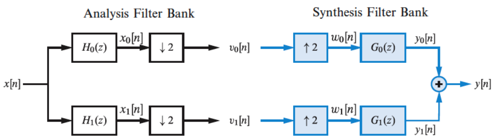
Polyphase Decimation



Polyphase Upsampling

Perfect reconstruction filters

Condition for perfect reconstruction: $y[n] = Gx[n-n_d] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) = Gz^{-n_d}X(z)$



Input and output of upper channel using $Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X(W_D^k z^{1/D})$ and $Y(z) = X(z^I)$

$$\begin{aligned} V_0(z) &= \tfrac{1}{2}H_0(z^{1/2})X(z^{1/2}) + \tfrac{1}{2}H_0(-z^{(1/2)})X(-z^{1/2}) \\ Y_0(z) &= V_0(z^2)G_0(z) \\ Y_0(z) &= \tfrac{1}{2}[H_0(z)X(z) + H_0(-z)X(-z)]G_0(z) \end{aligned}$$

Input and output of lower channel

$$\begin{aligned} V_1(z) &= \tfrac{1}{2}[H_1(z^{1/2})X(z^{1/2}) + H_1(-z^{1/2})X(-z^{1/2})] \\ Y_1(z) &= V_1(z^2)G_1(z) \\ Y_1(z) &= \tfrac{1}{2}[H_1(z)X(z) + H_1(-z)X(-z)]G_1(z) \end{aligned}$$

$$Y(z) = \tfrac{1}{2}[T(z)X(z) + A(z)X(-z)]$$

$$\underbrace{T(z) = H_0(z)G_0(z) + H_1(z)G_1(z)}_{Aliasing} \quad \underbrace{A(z) = H_0(-z)G_0(z) + H_1(-z)G_1(z)}_{phase/magnitude\ distortion}$$

$T(z) = H_0(z)G_0(z) + H_1(z)G_1(z) = Gz^{-n_d}$ $\quad A(z) = H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$

Perfect reconstruction condition:

Perfect reconstruction condition

$\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} Gz^{-n_d} \\ 0 \end{bmatrix}$ Setting $G = 2$ preserves the amplitude of input signal.

Synthesis filters

$G_0(z) = \frac{2z^{-n_d}}{\Delta_m(z)} H_1(-z),\ G_1(z) = -\frac{2z^{-n_d}}{\Delta_m(z)} H_0(-z),$
$\Delta_m(z) = H_0(z)H_1(-z) - H_0(-z)H_1(z) = -\Delta_m(-z)$

Perfect reconstruction conditions:

$\underbrace{z^{n_d} H_0(z)G_0(z)}_{R(z)} + \underbrace{z^{n_d} H_1(z)G_1(z)}_{R(-z)} = 2$ Since $H_1(z) = G_0(-z)\Delta_m(-z)\frac{1}{2}(-z)^{n_d}$

Polyphase decomposition of $R(z) = R_0(z^2) + z^{-1}R_1(z^2)$ with PR yields $R_0(z^2) + z^{-1}R_1(z^2) + R_0(z^2) - z^{-1}R_1(z^2) = 2$ or $R_0(z^2) = 1$ which makes $R(z)$ a half-band filter *i.e.* $\mathcal{F}^{-1}$[rectangular filter with cutoff $\pi/2$]

Design of a two-channel PR filter bank

- Obtain an $R(z)$ satisfying $R(z) = 1 + z^{-1}R_1(z^2)$
- perform the factorization $R(z) = z^{n_d} H_0(z)G_0(z)$
- assign the remaining filters from $G_0(z) = \frac{2z^{-n_d}}{\Delta_m(z)} H_1(-z), G_1(z) = -\frac{2z^{-n_d}}{\Delta_m(z)} H_0(-z)$

Quadrature mirror filter (QMF) banks

Design a low-pass filter $H(z)$ and then determine the filters in the bank as;
$H_0(z) = H(z),\ H_1(z) = H(-z)$
$G_0(z) = H(z),\ G_1(z) = -H(-z)$
Time and frequency domain characteristics :

$$
\begin{aligned}
h_0[n] = h[n] &\overset{DTFT}{\longleftrightarrow} H_0(\omega) = H(\omega) \\
h_1[n] = (-1)^n h[n] &\overset{DTFT}{\longleftrightarrow} H_1(\omega) = H(\omega - \pi) \\
g_0[n] = h[n] &\overset{DTFT}{\longleftrightarrow} G_0(\omega) = H(\omega) \\
g_1[n] = (-1)^{n+1} h[n] &\overset{DTFT}{\longleftrightarrow} H_1(\omega) = -H(\omega - \pi)
\end{aligned}
$$

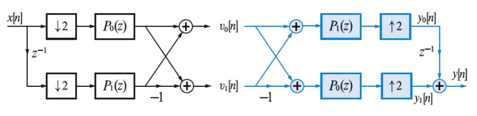
Since $H_0(\omega) = H_1(\omega - \pi)$, $\omega \longrightarrow \omega + \pi/2$, leads to $|H_0(\omega + \pi/2)| = |H_1(\pi/2 - \omega)|$
Symmetric about quadrature frequency $2\pi/4$

Perfect reconstruction property

$A(z) = H_0(-z)G_0(z) + H_1(-z)G_1(z) = H(-z)H(z) - H(z)H(-z) = 0$
$T(z) = H_0(z)G_0(z) + H_1(z)G_1(z) = H^2(z) - H^2(-z)$
$\Delta_m(z) = H_0(z)H_1(-z) - H_0(-z)H_1(z) = H^2(z) - H^2(-z)$

Polyphase decomposition

$H(z) = P_0(z^2) + z^{-1}P_1(z^2)$
$T(z) = \Delta_m(z) = 4z^{-1}P_0(z^2)P_1(z^2)$
If $P_0(z^2) = b_0 z^{-n_0}$ and $P_1(z^2) = b_1 z^{-n_1}$
then FIR PR filter is a pure delay
$H(z) = b_0 z^{2n_0} + b_1 z^{-(2n_1+1)}$ but it has not much of a practical value.



Power complementary filters

A low pass filter with linear phase (Type I or II) $H(\omega) = A(\omega)e^{-j\omega M/2}$ has even symmetry
$T(\omega) = H^2(\omega) - H^2(-\omega) = e^{-j\omega M}\left[|H(\omega)|^2 - (-1)^M|H(\omega-\pi)|^2)\right]$
$M = 0$ causes $T(\omega) = 0$ leading to severe amplitude distortion.
Choose M odd (Type II FIR)
$T(\omega) = e^{-j\omega M}\left[|H(\omega)|^2 + |H(\omega-\pi)|^2)\right] = e^{-j\omega M}\underbrace{\left[|H_0(\omega)|^2 + |H_1(\omega)|^2)\right]}_{1}$

$|H_0(\omega)|^2 + |H_1(\omega)|^2) = 1$

No exact solution other than $H(z) = b_0 z^{2n_0} + b_1 z^{-(2n_1+1)}$

An approximate solution is obtained by

$$J = \alpha \underbrace{\int_{\omega_s}^{\pi} |H(\omega)|^2 d\omega}_{aliasing} + \underbrace{(1-\alpha)\int_{0}^{\pi} (1 - |T(\omega)|^2) d\omega}_{perfect\ reconstruction},\ 0 < \alpha < 1$$