Laplace Transform

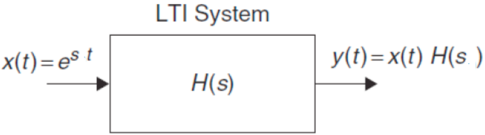
Lecture Notes

Ahmet Ademoglu, *PhD*
Bogazici University
Institute of Biomedical Engineering

Some concepts and illustrations in this lecture are adapted from the textbook,
Signals and Systems, 2nd Edition by Alan Oppenheim, Alan Willisky and H. Nawab, *Prentice Hall*.

Eigenfunctions of a LTI System

$s = \sigma + j\omega$ any number in Complex Plane



$$y(t) = \int_{-\infty}^{\infty} e^{s(t-\tau)} h(\tau) d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = e^{st} H(s)$$

Laplace Transform

Laplace Transform of a general signal $x(t)$ is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

and denoted as

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

Example

Let the signal $x(t) = e^{-at}u(t)$ and $a > 0$,
the Laplace Transform is

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-at}e^{-st}dt$$

$$X(s) = \frac{1}{s+a} \qquad Re\{s\} > -a$$

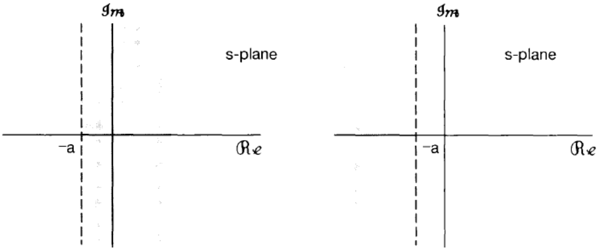
Example

Let the signal $x(t) = -e^{-at}u(-t)$ and $a > 0$,
the Laplace Transform is

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt = -\int_{-\infty}^{0} e^{-at}e^{-st}dt$$

$$X(s) = \frac{1}{s+a} \qquad\qquad Re\{s\} < -a$$

Region of Convergence



Example

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}$$

with ROC : $Re\{s\} > -1$

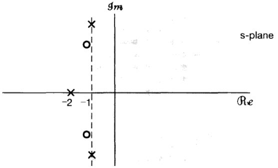
Example

$$x(t) = e^{-2t}u(t) + e^{-t}cos(3t)u(t)$$

$$x(t) = \left[e^{-2t} + \frac{1}{2}e^{-(1-3j)t} + \frac{1}{2}e^{-(1+3j)t}\right]u(t)$$

$$X(s) = \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)(s + 2)}$$

with ROC : $Re\{s\} > -1$

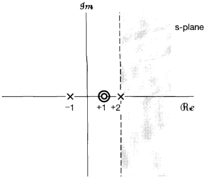


Example

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{2}e^{2t}u(t)$$

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} = \frac{(s-1)^2}{(s+1)(s-2)}$$

with ROC : $Re\{s\} > 2$



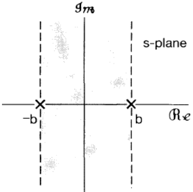
Example

$$x(t) = e^{-b|t|} \qquad and \qquad b > 0$$

$$X(s) = \int_{-\infty}^{0} e^{bt} e^{-st} dt + \int_{0}^{\infty} e^{-bt} e^{-st} dt$$

$$X(s) = \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2 - b^2}$$

with ROC : $-b < Re\{s\} < b$



PROPERTIES of LAPLACE TRANSFORM

If the Laplace transform of x(t) is rational *i.e.* X(s)= A(s)/B(s), then

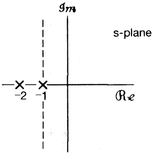
- if x(t) is right sided, the ROC is the region in the s-plane to the right of the rightmost pole.
- If x(t) is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

Example : Inverse Laplace Transform

$$X(s) = \frac{1}{(s+1)(s+2)}$$

with ROC : $Re\{s\} > -1$

$$X(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$



$$x(t) = (e^{-t} - e^{-2t})u(t)$$

Example : Inverse Laplace Transform

$$X(s) = \frac{1}{(s+1)(s+2)}$$

with ROC : $Re\{s\} < -2$

$$x(t) = (-e^{-t} + e^{-2t})u(-t)$$

Example : Inverse Laplace Transform

$$X(s) = \frac{1}{(s+1)(s+2)}$$

with ROC : $-2 < Re\{s\} < -1$

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$$

Properties of Laplace Transform

1. Linearity $\longrightarrow \alpha_1 x_1(t) + \alpha_2 x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} \alpha_1 X_1(s) + \alpha_2 X_2(s)$
   with ROC $=$ ROC of $X_1(s) \cap$ ROC of $X_2(s)$
2. Time Shifting $\longrightarrow x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s)$
   with ROC $=$ ROC of $X(s)$
3. Modulation $\longrightarrow e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s - s_0)$
   with ROC $=$ ROC of $X(s) + Re\{s_0\}$
4. Time Scaling $\longrightarrow x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X(s/a)$
   with ROC $=$ ROC of $X(s) * a$
5. Conjugation $\longrightarrow x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*)$
   with ROC $=$ ROC of $X(s)$

Properties of Laplace Transform

1. Convolution $\longrightarrow x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)X_2(s)$
   with ROC = ROC of $X_1(s) \cap$ ROC of $X_2(s)$
2. Differentiation in time $\longrightarrow \frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$
   with ROC = ROC of $X(s)$
3. Integration $\longrightarrow \int\limits_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s)$
   with ROC = ROC of $X(s) \cap \{Re\{s\} > 0\}$
4. Differentiation in s $\longrightarrow -tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds}$
   with ROC = ROC of $X(s)$
   Example: $te^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^2} \qquad Re\{s\} > -a$
   In general : $t^{n-1}/(n-1)!e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^n} \qquad Re\{s\} > -a$

Example

$$X(s) = \frac{2s^2 + 5s + 5}{(s+1)^2(s+2)} \qquad Re\{s\} > -1$$

$$X(s) = \frac{2}{(s+1)^2} - \frac{1}{s+1} + \frac{3}{s+2} \qquad Re\{s\} > -1$$

$$x(t) = \left[2te^{-t} - e^{-t} + 3e^{-2t}\right] u(t)$$

Analysis of LTI Systems using Laplace Transform

Causality

The ROC associated with the system function for a causal system is a right-half plane.

For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.

Examples:

1. $h(t) = e^{-t}u(t)$
   $H(s) = \frac{1}{s+1}$, $Re\{s\} > -1$
2. $h(t) = e^{-|t|}$
   $H(s) = \frac{-2}{s^2-1}$, $-1 < Re\{s\} < 1$
3. $H(s) = \frac{e^s}{s+1}$, $Re\{s\} > -1$
   $h(t) = e^{-(t+1)}u(t+1)$

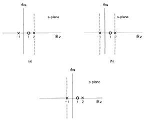
Analysis of LTI Systems using Laplace Transform

Stability

An LTI system is stable if and only if the ROC of its system function $H(s)$ includes the entire $j\omega$-axis [i.e., $Re\{s\} = 0$].

Example: $H(s) = \frac{s-1}{(s+1)(s-2)}$, no ROC is specified.

1. If the system is known to be causal, $h(t) = (\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t})u(t)$
2. If the system is known to be stable, $h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$
3. If the system is known to be unstable and anticausal, $h(t) = -(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t})u(-t)$



Analysis of LTI Systems using Laplace Transform

Stability of a Causal System

A causal system with rational system function H(s) is stable if and only if all of the poles of H(s) lie in the left-half of the s-plane-i.e., all of the poles have negative real parts.

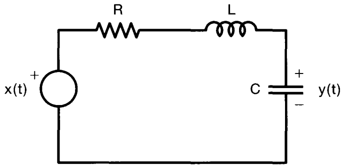
Examples:

$h(t) = e^{2t}u(t)$, unstable.

$H(s) = \frac{1}{s-2}$, $Re\{s\} > 2$

LTI Systems Characterized by Linear Differential Equations

$$RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t) = x(t)$$



$$RCsY(s) + LCs^2Y(s) + Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1/LC}{s^2 + R/Ls + (1/LC)}$$

Unilateral Laplace Transform

$$\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

Differentiation Property

$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{UL}} sX(s) - x(0^-)$

$\frac{d^2x(t)}{dt^2} \xleftrightarrow{\mathcal{UL}} s^2X(s) - x^{'}(0^-) - sx(0^-)$

Example

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

with $y^{'}(0^-) = 2, y(0^-) = 1,$
$s^2\mathcal{Y}(s) - sy(0^-) - y'(0^-) + 3(s\mathcal{Y}(s) - y(0^-)) + 2\mathcal{Y}(s) = \mathcal{X}(s)$
$\mathcal{Y}(s)(s^2 + 3s + 2) - (s + 5) = \mathcal{X}(s)$
If the input $x(t) = 2u(t)$ what is the output $y(t)$?
$\mathcal{X}(s) = \frac{2}{s} \longrightarrow \mathcal{Y}(s)(s^2 + 3s + 2) = \frac{2}{s} + (s + 5)$
$\mathcal{Y}(s) = \frac{s^2+5s+2}{s(s+1)(s+2)} = -\frac{2}{s+2} + \frac{2}{s+1} + \frac{1}{s}$

```
[R,P]= residue ([1 5 2],poly([0 -1 -2]))
R = [-2 2 1]
P = [-2 -1 0]
```

$$\mathcal{Y}(s) = -\frac{2}{s+2} + \frac{2}{s+1} + \frac{1}{s} \leftrightarrow y(t) = -2e^{-2t} + 2e^{-t} + u(t)$$

Initial and Final Value Theorems

- Initial Value Theorem $\longrightarrow x(0^+) = \lim_{s\to\infty} sX(s)$

$$\int_{0^-}^{\infty} \frac{d}{dx}x(t)e^{-st}dt\Big|_{s=\infty} = sX(s)\Big|_{s=\infty} - x(0^-) =$$

$$\int_{0^-}^{0^+} \frac{d}{dx}x(t)e^{-st}dt\Big|_{s=\infty} + \int_{0^+}^{\infty} \frac{d}{dx}x(t)e^{-st}dt\Big|_{s=\infty}$$

$$= x(0^+) - x(0^-) + 0$$

- Final Value Theorem $\longrightarrow x(\infty) = \lim_{s\to 0} sX(s)$

$$\int_{0^-}^{\infty} \frac{d}{dx}x(t)e^{-st}dt\Big|_{s=0} = sX(s)\Big|_{s=0} - x(0^-) = x(\infty) - x(0^-)$$