Infinite Impulse Response (IIR) filters

Lecture Notes

Ahmet Ademoglu, *PhD*
Bogazici University
Institute of Biomedical Engineering

Some concepts and illustrations in this lecture are adapted from the textbooks,
Applied Digital Signal Processing, Dimitris G. Manolakis & K. Ingle, *Cambridge*.

IIR versus FIR Filters

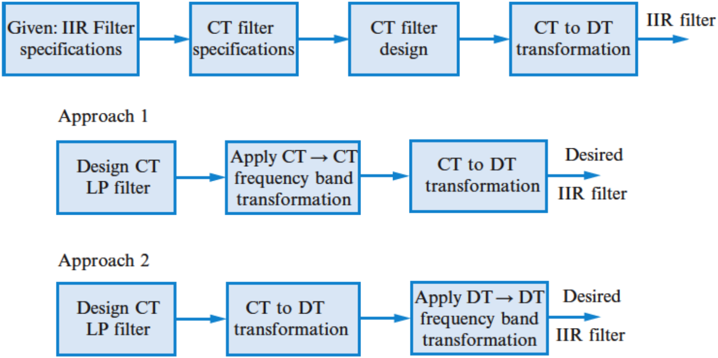
IIR Advantages

- Lower complexity (lower order)
- Lower output delay
- Easier computations for design

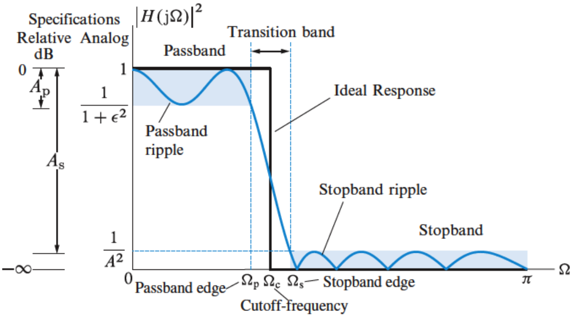
FIR advantages

- Linearity in Phase
- Flexibility in frequency response
- Realization in hardware
- Finite duration transients

Design of IIR Filters



Analog Lowpass Filter



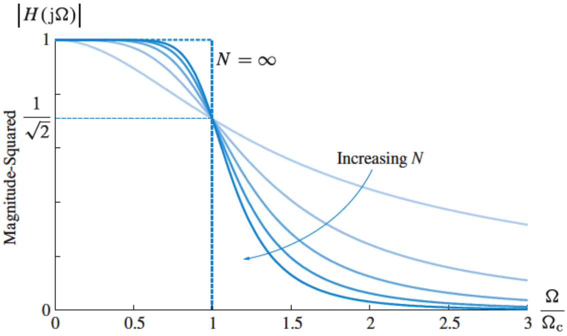
$$|H_d(\Omega)|^2 = \begin{cases} 1 & 0 \leq |\Omega| \leq \Omega_c, \\ 0 & |\Omega| > \Omega_c \end{cases}$$

$$|H_c(\Omega)|^2 = \frac{1}{1 + V^2(\Omega)}$$

Butterworth Approximation

$$|H_b(\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}, \; N = 1, 2 \ldots$$

Maximally flat approximation



Design Procedure

$$\frac{1}{1+(\Omega_p/\Omega_c)^{2N}} \geq \frac{1}{1+\epsilon^2}$$

and

$$\frac{1}{1+(\Omega_s/\Omega_c)^{2N}} \leq \frac{1}{1A^2}$$

which yields

$$\frac{\Omega_s^{2N}}{A^2-1} \geq \Omega_c^{2N} \geq \Omega_p^{2N}\frac{1}{\epsilon^2}$$

$$N = \frac{1}{2}\frac{\log_{10}\left[(A^2-1)/\epsilon^2\right]}{\log(\Omega_s/\Omega_p)}$$

Example: Determine the order N of Butterworth LPF with an attennuation of 1 dB at 1 kHz and 40 dB at 5 kHz.

$$10\log_{10}\left(\frac{1}{1+\epsilon^2}\right) = -1 \longrightarrow \epsilon^2 = 0.2589$$

$$10\log_{10}\left(\frac{1}{A^2}\right) = -40 \longrightarrow A^2 = 10^4$$

$$N = \frac{1}{2}\frac{\log_{10}\left[(A^2-1)/\epsilon^2\right]}{\log(\Omega_s/\Omega_p)} = \frac{\log_{10}(196.5)}{\log_{10}(5)} = 3.28 \approx 4$$

$$\frac{\Omega_s^{2N}}{A^2-1} \geq \Omega_c^{2N} \geq \Omega_p^{2N}\frac{1}{\epsilon^2} \longrightarrow \frac{5.10^3}{(10^4-1)^{1/8}} \geq \Omega_c \geq 1.10^3\frac{1}{0.2589^{1/8}}$$

$$\mathbf{1581} \;\; \geq f_c \geq 1184$$

Design in MATLAB

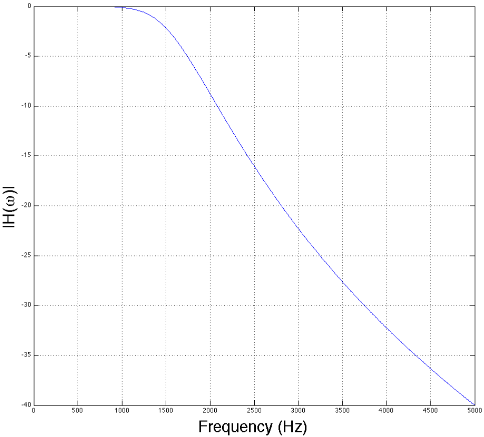
Version 1

```
[N,Omegac] = buttord(2*pi*1000,2*pi*5000,1,40,'s');
[B,A]=butter(N,Omegac,'s');
omega= [0:0.1:5]*2*pi*1000;
H = freqs(B,A,omega);
plot(omega/2/pi,20*log10(abs(H)));grid
```

Version 2

```
[N,Omegac] = buttord(1,5,1,40,'s');
[B,A]=butter(N,Omegac,'s');
omega= [0:0.1:5];
H = freqs(B,A,omega);
plot(omega*1000,20*log10(abs(H)));grid
```

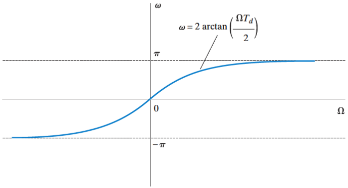
Frequency Response

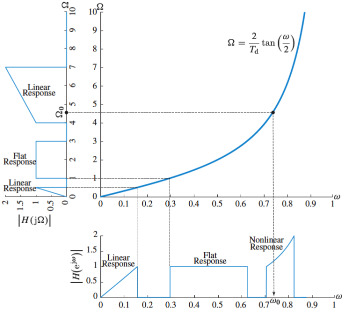


Bilinear Transformation of Analog to Digital Domain

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

If $T = 2$ then $j\Omega = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \tan\frac{\omega}{2} \longrightarrow \Omega = \tan\frac{\omega}{2}$





Design of Digital IIR LPF

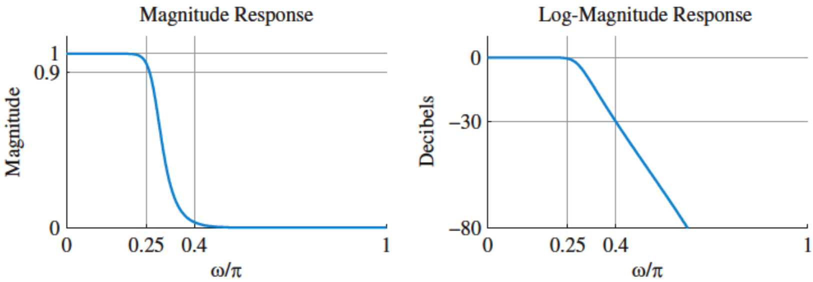
Given the Digital Filter Specifications

1. Normalize the frequencies by the sampling rate
2. Prewarp the frequencies
3. Design the low pass filter
4. Apply bilinear transform to convert the analog filter into digital

Example: $\omega_p = 0.25\pi$, $\omega_s = 0.4\pi$, $A_p = 1$ dB, $A_s = 30$ dB

Prewarp $\rightarrow \Omega_p = \tan\frac{\omega}{2} = \tan\frac{0.25\pi}{2} = 0.41$, $\Omega_s = \tan\frac{0.4\pi}{2} = 0.73$

```
[N,Omegac]=buttord(0.41,0.73,1,30,'s');
[D,C] = butter(N,Omegac,'s');
[B,A] = bilinear(D,C,1/2);
omega= [0 : 0.01 : 0.62];
H = freqz(B,A,omega*pi);
plot(omega,20*log10(abs(H)));
```


Magnitude Response
Magnitude
1
0.9
0
0
0.25
0.4
1
ω/π
Log-Magnitude Response
Decibels
0
−30
−80
0
0.25
0.4
1
ω/π

Design of Digital IIR HPF

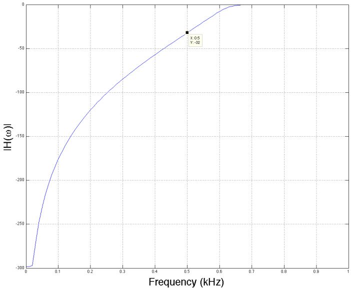
Given the Digital Filter Specifications

1. Normalize the frequencies by the sampling rate
2. Prewarp the frequencies
3. Apply HP to LP frequency transformation $\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$
4. Design the low pass filter
5. Apply LP to HP transformation by $s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}}$
6. Apply bilinear transform to convert the analog filter to digital

Example: $F_T = 2kHz$, $f_p = 700Hz$, $f_s = 500Hz$, $A_p = 1$ dB, $A_s =$32 dB

Normalize the frequencies $\omega_p = \frac{2\pi f_p}{F_T} = 0.7\pi$, $\omega_s = \frac{2\pi f_s}{F_T} = 0.5\pi$ Prewarp $\rightarrow \hat{\Omega}_p = \tan\frac{0.7\pi}{2} = 1.96$, $\hat{\Omega}_s = \tan\frac{0.5\pi}{2} = 1.0$ Choose $\Omega_p = 1$ then $\Omega_s = \frac{1\times 1.96}{1.0}$

```
[N,Omegac]=buttord(1,1.96,1,32,'s');
[D,C] = butter(N,Omegac,'s');
[num,den] =  lp2hp( D,C,1.96);
[B,A] = bilinear(num,den,1/2);
omega= [0 : 0.01 : 1];
H = freqz(B,A,omega*pi);
plot(omega,20*log10(abs(H)));
```



Design of Digital IIR BPF

Given the Digital Filter Specifications

1. Normalize the frequencies by the sampling rate
2. Prewarp the frequencies
3. Apply BP to LP frequency transformation:
   $\Omega = -\Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})}$
4. Design the low pass filter
5. Apply LP to BP transformation by $s = \Omega_p \frac{\hat{s}^2 + \hat{\Omega}_0^2}{\hat{s}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})}$
6. Apply bilinear transform to convert the analog filter to digital

Design of Digital IIR BPF

Example: $\omega_{p1} = 0.45\pi$, $\omega_{p1} = 0.65\pi$, $\omega_{s1} = 0.3\pi$, $\omega_{s2} = 0.75\pi$, $A_p = 1$ dB, $A_s = 40$ dB

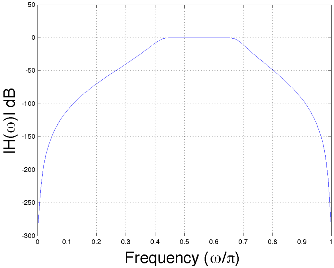
Adjust the frequencies

- $\hat{\Omega}_{p1} = \tan(0.45\pi/2) = 0.85$,
- $\hat{\Omega}_{p2} = \tan(0.65\pi/2) = 1.63$,
- $\hat{\Omega}_{s1} = \tan(0.3\pi/2) = 0.51$,
- $\hat{\Omega}_{s2} = \tan(0.75\pi/2) = 2.41$
- $\hat{\Omega}_{s1}\hat{\Omega}_{s2} \leftrightarrow \hat{\Omega}_{p1}\hat{\Omega}_{p2}, \hat{\Omega}_{s2} = \hat{\Omega}_{p1}\hat{\Omega}_{p2}/\hat{\Omega}_{s1} = 1.39/0.51 = 2.73$,
- $B_w = \hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 0.77$, $\hat{\Omega}_0 = (\hat{\Omega}_{p1}\hat{\Omega}_{p2})^{1/2} = 1.18$

LPF Frequencies :

$\Omega_p = 1$, $\Omega_s = 1 \times \frac{1.39-0.51^2}{0.51\times0.77} = 2.86$

```
[N,Omegac]=buttord(1,2.86,1,40,'s');
[D,C] = butter(N,Omegac,'s');
[num,den] =  lp2bp( D,C,1.18,0.77);
[B,A] = bilinear(num,den,1/2 );
omega= [0 : 0.01 : 1];
H = freqz(B,A,omega*pi);
plot(omega,20*log10(abs(H)));
```

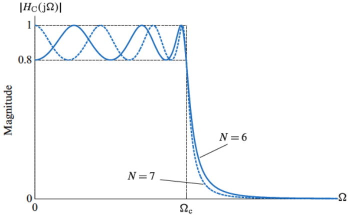


Chebysheff Polynomial I

$$\cos[(m+1)\theta] = 2\cos(\theta)\cos(m\theta) - \cos[(m-1)\theta],\ m \geq 1$$
$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x),\ m \geq 1 \text{ with } T_0(x) = 1 \text{ and } T_1(x) = x$$

$$T_N(x) = \cos(N\cos^{-1}x),\ |x| \leq 1$$
$$T_N(x) = \cosh(N\cosh^{-1}x),\ |x| > 1$$

$$|H_C(\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)}$$



$$\cosh(x) = \frac{e^x + e^{-x}}{2} \text{ and } \sinh(x) = \frac{e^x - e^{-x}}{2}$$

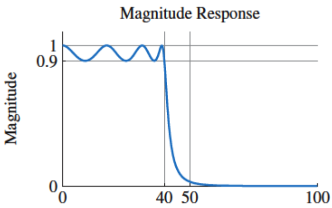
$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1}) \text{ and } \sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$$

$$\frac{1}{1 + \epsilon^2 T_N^2(\Omega_s/\Omega_p)} \leq \frac{1}{A^2},\ \cosh(N \cosh^{-1}(\Omega_s/\Omega_p)) \geq \frac{1}{\epsilon}\sqrt{A^2 - 1}$$

$$N \geq \frac{\ln(\beta + \sqrt{\beta^2 - 1})}{\ln(\alpha + \sqrt{\alpha^2 - 1})},\ \alpha = \frac{\Omega_s}{\Omega_p},\ \beta = \frac{1}{\epsilon}\sqrt{A^2 - 1}$$

Example: Design an analog LPF with $F_p = 40Hz$, $F_s = 50Hz$, $A_p = 1$ dB, $A_s = 30$ dB

```
[N,Omegac] = cheb1ord( 1, 50/40,1,30,'s');
[B,A] = cheby1(N,1,Omegac,'s');
omega= [0 :0.01:2];
H = freqs(B,A,omega);
plot(omega*40,20*log10(abs(H)));grid
```



Other Approximations

Inverse Chebyshev Approximation or Chebyshev (II):

$$|H_{IC}(\Omega)|^2 = 1 - \frac{1}{1 + \epsilon^2 T_N^2(\Omega_c/\Omega)}$$

$$|H_E(\Omega)|^2 = \frac{1}{1 + \epsilon^2 R_N^2(\Omega/\Omega_c)}$$

Elliptic Approximation

$$R_N(\Omega) = \begin{cases} \nu \frac{\Omega_1^2 - \Omega^2}{1 - \Omega_1^2 \Omega^2} \frac{\Omega_3^2 - \Omega^2}{1 - \Omega_3^2 \Omega^2} \cdots \frac{\Omega_{2N-1}^2 - \Omega^2}{1 - \Omega_{2N-1}^2 \Omega^2} & N \ even \\ \nu \Omega \frac{\Omega_2^2 - \Omega^2}{1 - \Omega_2^2 \Omega^2} \frac{\Omega_4^2 - \Omega^2}{1 - \Omega_4^2 \Omega^2} \cdots \frac{\Omega_{2N}^2 - \Omega^2}{1 - \Omega_{2N}^2 \Omega^2} & N \ odd \end{cases}$$