Fourier Series of Continuous Periodic Signals
Lecture Notes

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Some concepts and illustrations in this lecture are adapted from the textbook,
Signals and Systems, 2nd Edition by Alan Oppenheim, Alan Willisky and H. Nawab, *Prentice Hall*.

Continuous Fourier Series

Fourier Series Representations of Periodic Signals with period T,

$$x(t) = \sum_{k=\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

$\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency.
The a_k are determined by

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Specifically,

$$a_0 = \frac{1}{T} \int_T x(t) dt$$

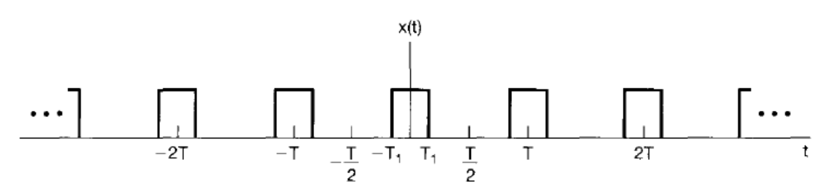
Example

$$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \frac{\pi}{4})$$

$$x(t) = 1 + \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} + e^{j\omega_0 t} + e^{-j\omega_0 t} + \frac{1}{2}e^{j2\omega_0 t}e^{\frac{j\pi}{4}} + \frac{1}{2}e^{-j2\omega_0 t}e^{\frac{-j\pi}{4}}$$

$$\begin{aligned}
a_0 &= 1, \\
a_1 &= (1 + \frac{1}{2j}) = a_{-1}^* \\
a_2 &= \frac{1}{2}e^{j\frac{\pi}{4}} = \frac{\sqrt{2}}{4}(1 + j) = a_{-2}^* \\
a_k &= 0, \qquad for\ |k| > 2
\end{aligned}$$

Example



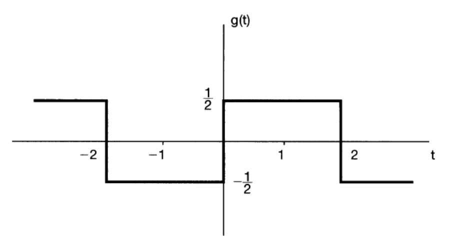
$$a_0 = \frac{1}{T}\int_{-T_1}^{T1} x(t)dt = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T1} x(t)e^{-jk\omega_0 t}dt = \frac{\sin(k\omega_0 T_1)}{k\pi} \qquad k \neq 0$$

Properties of Fourier Series

- Linearity $\rightarrow \alpha x_1(t) + \beta x_2(t) \overset{\mathcal{FS}}{\longleftrightarrow} \alpha a_k + \beta b_k$
- Time Shifting $\rightarrow x(t - t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k$
- Time Reversal $\rightarrow x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}$
- Time Scaling $\rightarrow x(\alpha t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ with period T/α
- Multiplication $\rightarrow x(t) \cdot y(t) \overset{\mathcal{FS}}{\longleftrightarrow} \sum_{m=-\infty}^{\infty} a_m b_{k-m}$
- Convolution $\rightarrow x(t) * y(t) \overset{\mathcal{FS}}{\longleftrightarrow} T a_k b_k,$
- Conjugation $\rightarrow x(t)^* \overset{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}$, for real $x(t)$ $a_k = a^*_k$
- Parseval's Relation $\rightarrow \frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=\infty}^{\infty} |a_k|^2$

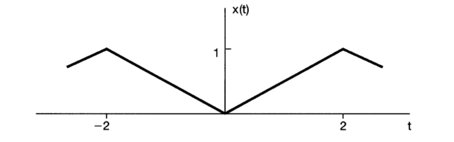
Example



$$g(t) = x(t-1) - 1/2$$

$$g_k = \begin{cases} \frac{(\sin \pi k/2)}{\pi k} e^{-jk\pi/2} & \textit{for} \quad k \neq 0, \\ 0 & \textit{for} \quad k = 0, \end{cases}$$

Example



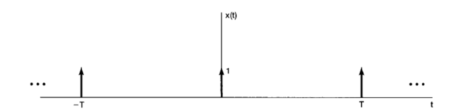
$$g(t) = \frac{dx(t)}{dt} \longrightarrow g_k = jk\omega_0 x_k$$

$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

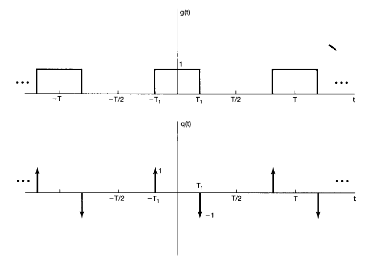
$$x_k = \frac{2(\sin \pi k/2)}{j(\pi k)^2} e^{-jk\pi/2} \qquad k \neq 0$$

$$x_0 = 2/4 = 1/2$$

Example



$$x(t) = \sum_{k=\infty}^{\infty} \delta(t - kT) \longrightarrow x_k = \frac{1}{T}$$



More Examples

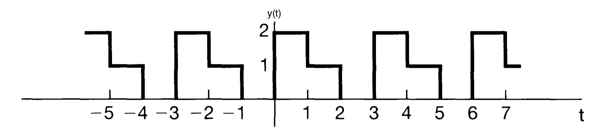
Example:
For the continuous-time periodic signal
$x(t) = 2 + \cos(\frac{2\pi}{3}t) + 4\sin(\frac{5\pi}{3}t)$,
determine the fundamental frequency ω_0 and the Fourier series coefficients x_k.

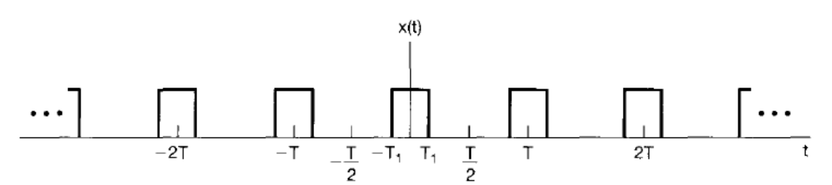
$$x(t) = 2 + \frac{1}{2}e^{j\frac{2\pi}{3}t} + \frac{1}{2}e^{-j\frac{2\pi}{3}t} + \frac{2}{j}e^{j\frac{5\pi}{3}t} - \frac{2}{j}e^{-j\frac{5\pi}{3}t}$$

$$\omega_0 = \frac{\pi}{3}$$

Example

Determine the Fourier Series of $x(t)$ with period T





$$y(t) = 2x(t-1/2) + x(t-3/2), y_k = 2x_k e^{-(jk\omega_0 1/2)} + x_k e^{(-jk\omega_0 3/2)}$$

$$x_k = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{\sin(k\pi/3)}{k\pi} \longrightarrow y_k = \frac{\sin(k\pi/3)}{k\pi}(2e^{-(jk\pi/3)} + e^{-jk\pi})$$

Example

Determine the signal $x(t)$ with period $T = 4$ from its Fourier series coefficients x_k

$$x_k = \begin{cases} jk & |k| < 3, \\ 0 & otherwise \end{cases}$$

$$x(t) = je^{j2\pi/4t} - je^{-j2\pi/4t} + 2je^{j2\pi/4*2t} - 2je^{-j2\pi/4*2t}$$

$$x(t) = -2\sin(\pi/2t) - 4\sin(\pi t)$$

Fourier Series and the LTI Systems

$$e^{j\omega t} \longrightarrow H(\omega)e^{j\omega t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t} \longrightarrow \sum_{k=-\infty}^{\infty} x_k H(k\omega_0) e^{jk\omega_0 t}$$

Example

Consider a continuous-time LTI system with impulse response

$$h(t) = e^{-4|t|}$$

Find the Fourier series representation of the output $y(t)$ for each of the following inputs:

i) $x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t-n)$

ii) $x(t)$ is the periodic wave given as below;