Finite Impulse Response (FIR) FILTERS

Lecture Notes

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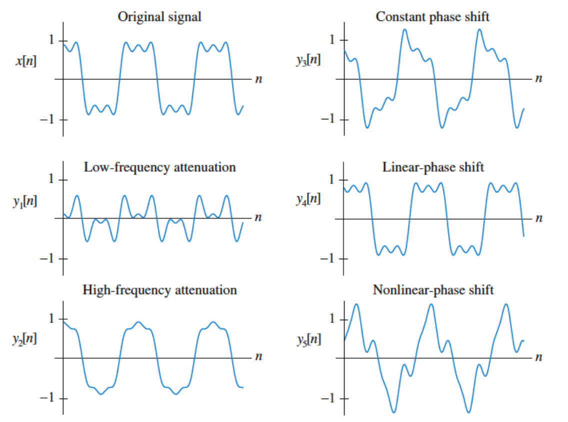
Some concepts and illustrations in this lecture are adapted from the textbook,

Applied Digital Signal Processing, Dimitris G. Manolakis & K. Ingle, *Cambridge*.

Effect of a LTI system $H(e^{j\omega})$ on magnitude and phase of a signal



Magnitude Distortion Phase Distortion

Phase and Group Delay

$$A\cos(\omega_0 n + \phi) \xrightarrow{LTI} A|H(\omega_0)|\cos(\omega_0[n + \tfrac{\theta(\omega_0)}{\omega_0}] + \phi)$$

Phase Delay : $\tau_{pd}(\omega) = -\frac{\theta(\omega)}{\omega}$ which is a time delay at ω.

$$A\cos(\omega_0 n)\cos(\omega_c n) = \tfrac{A}{2}\cos((\underbrace{\omega_c - \omega_0}_{\omega_l})n) + \tfrac{A}{2}\cos((\underbrace{\omega_c + \omega_0}_{\omega_u})n)$$

$$A\cos(\omega_0 n)\cos(\omega_c n) \xrightarrow{LTI} \tfrac{A}{2}\cos(\omega_l n + \theta(\omega_l)) + \tfrac{A}{2}\cos(\omega_u n + \theta(\omega_u))$$

$$= A\cos(\omega_c n + \frac{\theta(\omega_u) + \theta(\omega_l)}{2})\cos(\omega_0 n + \frac{\theta(\omega_u) - \theta(\omega_l)}{2})$$

assuming $|H(\omega)| \approx 1$ in the range $\omega_l \leq \omega \leq \omega_u$.

Output carrier and modulation frequencies are the same as those at the input but not the phases.

Phase Delay and Group Delay

If $\omega_0 << \omega_c$,

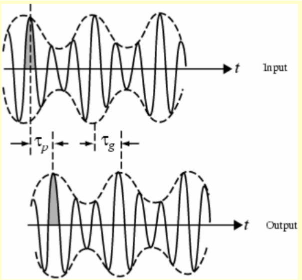
$$\theta(\omega_u) = \theta(\omega_c + \omega_0) \approx \theta(\omega_c) + \left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c} \omega_0$$

$$\theta(\omega_l) = \theta(\omega_c - \omega_0) \approx \theta(\omega_c) - \left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c} \omega_0$$

$-\frac{\theta(\omega_u)+\theta(\omega_l)}{2\omega_c} \approx -\frac{\theta(\omega_c)}{\omega_c}$ is the phase delay for the carrier signal.

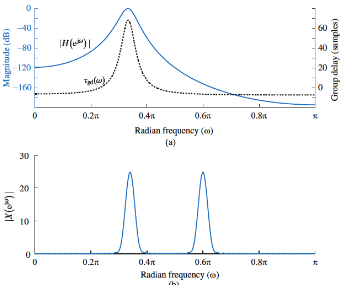
$-\frac{\theta(\omega_u)-\theta(\omega_l)}{2\omega_0} \approx -\frac{d\theta(\omega)}{d\omega} = \tau_{gd}(\omega)$ is the group delay or the envelope delay.

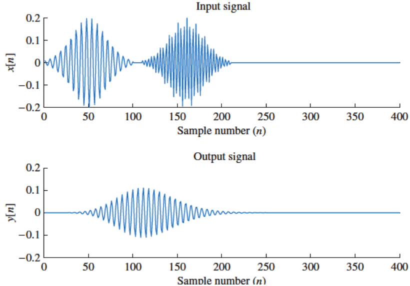
τ_{gd} : measure of the linearity of the phase and is the time delay between the input and output waveforms.



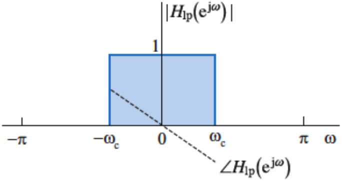
Example: $H(z) = \frac{b_0}{[1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}]^K}$, $r = 0.9$, $\omega_0 = \pi/3$, $K = 8$

Input signal : $s[n-n_1]\cos(0.34\pi n) + s[n-n_2]\cos(0.6\pi n)$
where $s(t) = \frac{1}{2\pi}\exp(-\frac{1}{2}\frac{(t-\mu)^2}{\sigma^2})$





Ideal FIR Filters with Linear Phase



$$H_{LP}(\omega) = \begin{cases} e^{-j\alpha\omega} & |\omega| < \omega, \\ 0 & \omega_c < |\omega| \leq \pi \end{cases} \longleftrightarrow h_{LP}[n] = \frac{\sin[\omega_c(n-\alpha)]}{\pi(n-\alpha)}$$

Types of Linear Phase FIR Transfer Functions

Example : $N = 8$, Symmetric FIR Filter

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[3]z^{-5} + h[2]z^{-6} + h[1]z^{-7} + h[0]z^{-8}$$

$$H(z) = [2h[0]\cos(4\omega) + 2h[1]\cos(3\omega) + 2h[2]\cos(2\omega) + 2h[3]\cos(\omega) + h[4]]e^{-j4\omega}$$

Type I : N even, symmetric $h[n] = h[N-n]$, $0 \leq n \leq N$

Phase: $\theta(\omega) = -4\omega + \beta$, $\beta = 0$, or π (inside [.] can be (-) as well)

$$\tau_{gd} = -\frac{d\theta(\omega)}{d\omega} = N/2$$

$$\breve{H}(\omega) = h[N/2] + 2\sum_{n=1}^{N/2} h[N/2-n]\cos[\omega n] = \sum_{n=0}^{N/2} a[k]\cos(\omega n)$$

$$H(\omega) = e^{-jN\omega/2}\breve{H}(\omega)$$

Type II : N odd, symmetric $h[n] = h[N-n]$, $0 \leq n \leq N$

Phase : $\theta(\omega) = -N/2\omega + \beta$, $\beta = 0$, or π

$$\tau_{gd} = -\frac{d\theta(\omega)}{d\omega} = N/2$$

$$\breve{H}(\omega) = 2\sum_{n=1}^{(N+1)/2} h[\tfrac{N+1}{2} - n]\cos[\omega(n - \tfrac{1}{2})] = \cos(\tfrac{\omega}{2})\sum_{n=0}^{(M-1)/2} b[n]\cos(\omega n)$$

$$H(\omega) = e^{-jN\omega/2}\breve{H}(\omega)$$

Type III : N even, antisymmetric $h[n] = -h[N-n]$, $0 \leq n \leq N$

Phase : $\theta(\omega) = \pi/2 - N/2\omega + \beta$, $\beta = 0$, or π

$$\tau_{gd} = -\frac{d\theta(\omega)}{d\omega} = N/2$$

$$\breve{H}(\omega) = 2\sum_{n=1}^{N/2} h[N/2 - n]\sin[\omega(n)] = \sin(\omega)\sum_{n=0}^{M/2} c[n]\cos(\omega n)$$

$$H(\omega) = je^{-jN\omega/2}\breve{H}(\omega)$$

Type IV : N odd, antisymmetric $h[n] = -h[N-n]$, $0 \leq n \leq N$

Phase : $\theta(\omega) = \pi/2 + -N/2\omega + \beta$, $\beta = 0$, or π

$\tau_{gd} = -\frac{d\theta(\omega)}{d\omega} = N/2$

$$\breve{H}(\omega) = 2\sum_{n=1}^{(N+1)/2} h[(N+1)/2 - n]\sin[\omega(n-1/2)] = \sin(\tfrac{\omega}{2})\sum_{n=0}^{(M-1)/2} d[k]\cos(\omega n)$$

$$H(\omega) = je^{-jN\omega/2}\breve{H}(\omega)$$

Amplitude of FIR filter transfer functions

$$\breve{H}(\omega) = Q(\omega)P(\omega),$$

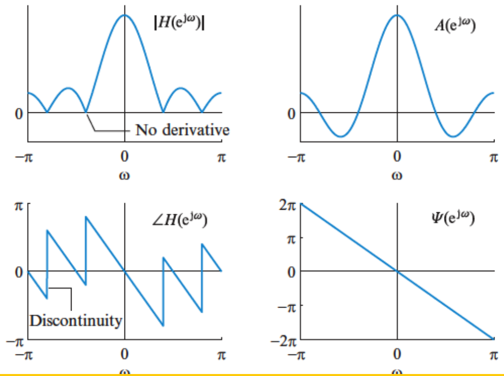
$$P(\omega) = \sum_{k=0}^{R} p[k]\cos(\omega k)$$

$$Q(\omega) = \begin{cases} 1 & \textit{Type I} \\ \cos(\omega/2) & \textit{Type II} \\ \sin(\omega) & \textit{Type III} \\ \sin(\omega/2) & \textit{Type IV} \end{cases}$$

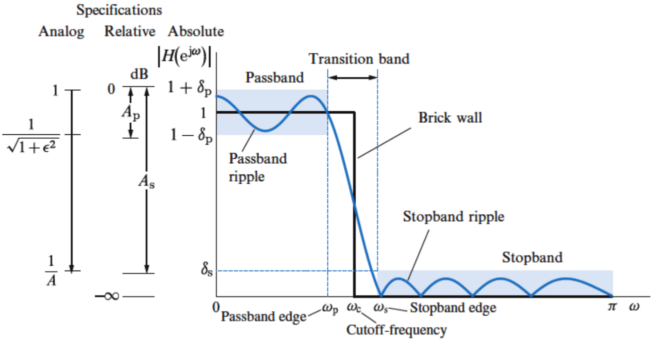
Phase and Amplitude Response of FIR Filters (I-IV)

$$H(\omega) = |H(\omega)|e^{-j\alpha\omega} = A(\omega)e^{j\omega\alpha+j\beta} = A(\omega)e^{j\Psi(\omega)}$$

The group delay is $\tau_{gd} = -\frac{d\psi(\omega)}{d\omega}$



Optimality criteria for filter design



Desired frequency response

$$H_d(\omega) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n}$$

Mean squared error

$$E_2 = \left[\frac{1}{2\pi} \int_B \left| H_d(e^{j\omega}) - H(e^{j\omega}) \right|^2 d\omega \right]^{1/2}$$

Minimax approximation

$$E_\infty = \max_{\omega \in B} |H_d(e^{j\omega}) - H(e^{j\omega})|$$

Maximally flat approximation

$A_d(\omega) = A_d(\omega_0) + \frac{A_d(\omega_0)^{(1)}}{1!}(\omega - \omega_0) + \frac{A_d(\omega_0)^{(2)}}{2!}(\omega - \omega_0)^2 + \ldots$

$A(\omega) = A(\omega_0) + \frac{A(\omega_0)^{(1)}}{1!}(\omega - \omega_0) + \frac{A(\omega_0)^{(2)}}{2!}(\omega - \omega_0)^2 + \ldots$

$A_d(\omega_0) = A(\omega_0)$

$A_d^{(i)}(\omega_0) = A^{(i)}(\omega_0)$ for $i = 1, 2, \ldots, m-1$

$$E(\omega) = A_d(\omega) - A(\omega) = \frac{A^{(m)}(\omega_0) - A_d^{(m)}(\omega_0)}{m!}(\omega - \omega_0)^m + \ldots$$

Mean Squared Error

$$E_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_d(e^{j\omega}) - H(e^{j\omega}) \right|^2 d\omega$$

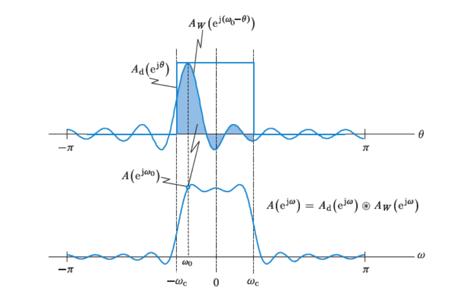
$$\epsilon^2 = \sum_{n=0}^{M} [h_d[n] - h[n]]^2 + \sum_{n=-\infty}^{-1} h_d[n]^2 + \sum_{n=M+1}^{\infty} h_d[n]^2$$

The optimum solutions is

$$h[n] = \begin{cases} h_d[n] & 0 \leq n \leq M, \\ 0 & otherwise \end{cases}$$

$$w[n] == \begin{cases} 1 & 0 \leq n \leq M, \\ 0 & otherwise \end{cases} \quad h[n] = h_d[n]w[n]$$

$$W[\omega] = \frac{\sin(\omega(M+1)/2)}{\sin(\omega/2)} e^{-j\omega M/2}$$



The width of the main lobe $= \pm 2\pi/(M+1)$
The first negative side lobe $= \pm 3\pi/(M+1)$
$= |A_W(3\pi/(M+1))/A_W(0)| = |\sin(\frac{3\pi}{2})/\sin(\frac{3\pi}{2(M+1)})/(M+1)| \approx \frac{2}{3\pi} = -13$ dB

Gibbs Phenomenon

The properties of $A(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} A_d(\theta)A_w(\omega-\theta)d\theta$ at the vicinity of ω_c :

For $\omega < \omega_c$ $A(\omega) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} A_w(\omega-\theta)d\theta = \frac{1}{2\pi}[\int_{-\omega_c}^{\omega} + \int_{\omega}^{\omega_c}]A_W(\omega-\theta)d\theta$

$\phi = \omega - \theta \rightarrow \frac{1}{2\pi}\int_{-\omega_c}^{\omega} A_W(\omega-\theta)d\theta = \frac{1}{2\pi}\int_{0}^{\omega+\omega_c} A_W(\phi)d\phi$

$\phi = (\omega-\theta)(M+1)/2 \rightarrow \frac{1}{2\pi}\int_{\omega}^{\omega_c} A_W(\omega-\theta)d\theta = \frac{-1}{(M+1)\pi}\int_{0}^{(\omega-\omega_c)(M+1)/2} A_W(\frac{2\phi}{(M+1)})d\phi$

$\phi = -\alpha, \quad \frac{1}{(M+1)\pi}\int_{0}^{(\omega_c-\omega)(M+1)/2} A_W(\frac{2\alpha}{(M+1)})d\alpha$

$A(\omega) \approx \frac{1}{2} + \frac{1}{\pi}\int_{0}^{(\omega_c-\omega)(M+1)/2} \frac{\sin(\phi)}{\phi}d\phi$ since $A_W(x) = A_W(-x)$.

$Si(\xi) = \int_{0}^{\xi} \frac{\sin(\phi)}{\phi}d\phi,$

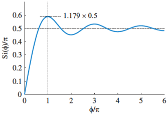
$A_W(\frac{2\phi}{(M+1)}) \approx \frac{\sin(\phi)}{\phi/(M+1)}$

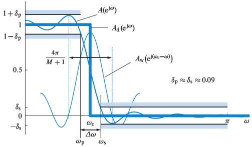
$A(\omega) \approx \frac{1}{2} + \frac{1}{\pi}Si[(\omega_c-\omega)(M+1)/2]$ for $\omega < \omega_c$

$$A(\omega) \approx \frac{1}{2} + \frac{1}{\pi} Si[(\omega - \omega_c)(M+1)/2] \quad \text{for } \omega < \omega_c$$
$$A(\omega) \approx \frac{1}{2} - \frac{1}{\pi} Si[(\omega - \omega_c)(M+1)/2] \quad \text{for } \omega > \omega_c$$

$$\begin{array}{lllllll} \omega & = & \omega_c & & A(\omega) & = & 0.5 \\ \omega & = & \omega_c - 2\pi/(M+1) & & A(\omega) & = & 0.5 + Si(\pi)/\pi \approx 1.0895 \\ \omega & = & \omega_c + 2\pi/(M+1) & & A(\omega) & = & 0.5 - Si(\pi)/\pi \approx 0.0895 \end{array}$$

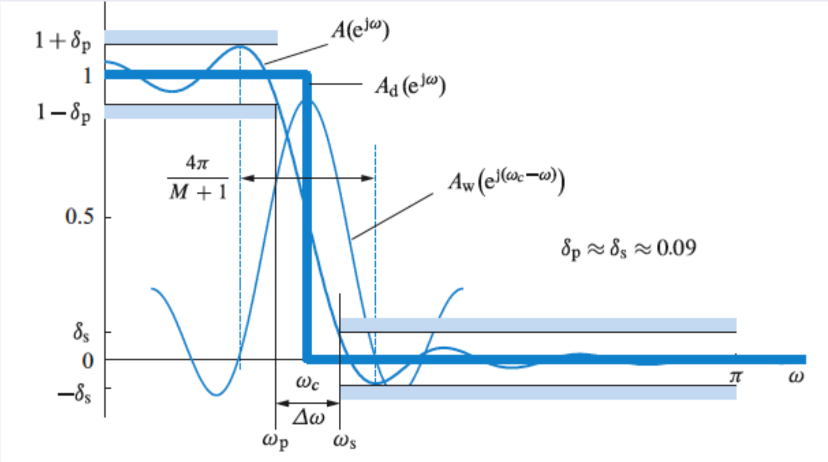
- $\delta_p \approx \delta_s \approx 0.0895 \rightarrow$ irrespective of the order M
- $20\log_{10}(1.0895) = 0.75$dB and $20\log_{10}(0.0895) = -21$dB $\rightarrow$not sufficient
- ω_p or (ω_s) is determined by finding the ξ value for which the $Si(\xi)$ reaches to the level of $0.5 - \delta_p$ which is 0.46π from which $\omega = \frac{2\xi}{M+1} = \frac{0.9\pi}{M+1}$.
  $\Delta\omega = \omega_s - \omega_p \approx \frac{1.8\pi}{M} \leq \frac{4\pi}{M+1} \rightarrow$ transition band depends on the order M



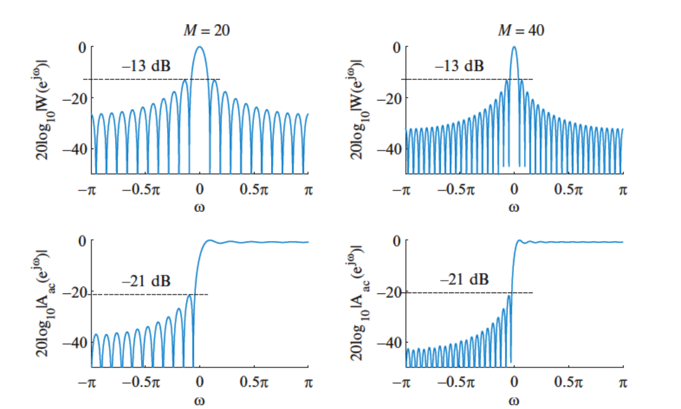



```
d=0.01; X=20/pi;
t =[d:d:X];
I=cumsum(sin(t)./t)*d/pi;
t(max(find(I<(.5-.0895))))/pi % =0.46;
```

Gibbs Phenomenon



As the order increases, the highest side lobe and minimum stopband attenuation remains the same. Only the transition band decreases.



Alternative Window Functions: Smoother Impulse Response

- Bartlett (Triangular) $w[n] = \begin{cases} 2n/M & 0 \le n \le M/2, M \text{ even} \\ 2 - 2n/M & M/2 < n \le M \\ 0 & \text{otherwise} \end{cases}$
- Hann
$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M) & 0 \le n \le M, \\ 0 & \text{otherwise} \end{cases}$$
- Hamming
$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M) & 0 \le n \le M, \\ 0 & \text{otherwise} \end{cases}$$
- Blackman
$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M) & 0 \le n \le M, \\ 0 & \text{otherwise} \end{cases}$$

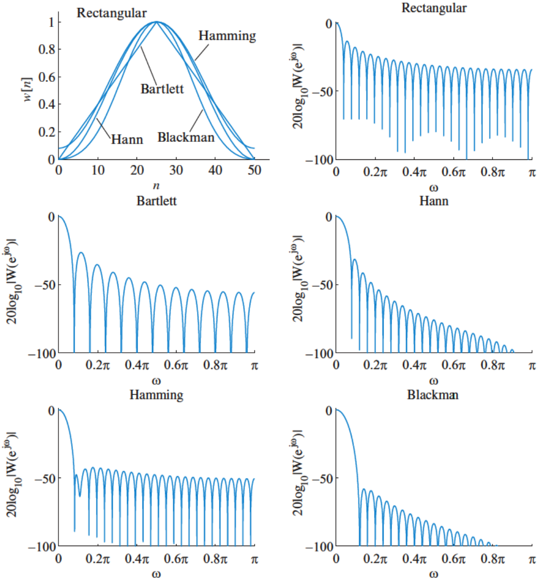

Rectangular
Hamming
Bartlett
Hann
Blackman
w[n]
n
Rectangular
Bartlett
Hann
Hamming
Blackman
$20\log_{10}|W(e^{j\omega})|$
ω

Table 10.3 Properties of commonly used windows ($L = M + 1$).

Window name	Side lobe level (dB)	Approx. $\Delta\omega$	Exact $\Delta\omega$	$\delta_p \approx \delta_s$	A_p (dB)	A_s (dB)
Rectangular	−13	$4\pi/L$	$1.8\pi/L$	0.09	0.75	21
Bartlett	−25	$8\pi/L$	$6.1\pi/L$	0.05	0.45	26
Hann	−31	$8\pi/L$	$6.2\pi/L$	0.0063	0.055	44
Hamming	−41	$8\pi/L$	$6.6\pi/L$	0.0022	0.019	53
Blackman	−57	$12\pi/L$	$11\pi/L$	0.0002	0.0017	74

Example: Design a lowpass linear-phase FIR filter with the following specifications: $\omega_p = 0.25\pi$, $\omega_s = 0.35\pi$, $A_p = -0.1$ dB, $A_s = 50$dB.

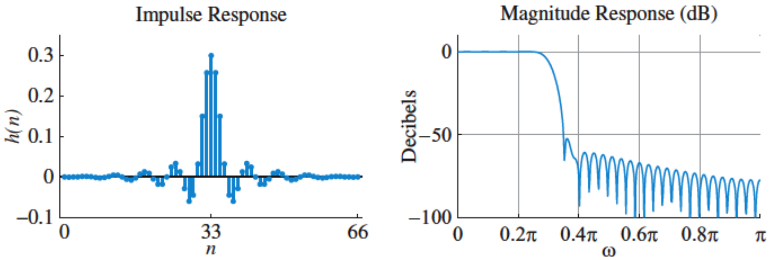
- $A_p = 20\log_{10}(\frac{1+\delta_p}{1-\delta_p}) = 0.1$, $\delta_p = 0.0058$
- $A_s = 20\log_{10}(\frac{1+\delta_p}{\delta_s}) = -50$, $\delta_s = 0.0032$.
- Set the ideal lowpass filter cutoff frequency $\omega_c = (\omega_p + \omega_s)/2 = 0.3\pi$, transition bandwidth, $\Delta\omega = \omega_s - \omega_p = 0.1\pi$.
- Hamming window provides at least 53 dB $>$ 50 dB attenuation which is greater than A = 50 dB.
- the transition bandwidth$\Delta\omega \approx 6.6\pi/L$, the minimum window length $L = 66$.
- Choose $L = 67$ or $M = 66$ to obtain a type-I filter.

The impulse response is $h[n] = w[n]h_d[n]$

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M) & 0 \leq n \leq M, \\ 0 & otherwise \end{cases}$$

$$h_d[n] = \frac{\sin[\omega_c(n-M/2)]}{\pi(n-M/2)}$$

```
wp = 0.25*pi; ws = 0.35*pi; Ap = 0.1; As = 50;
deltap = (10^(Ap/20)-1)/(10^(Ap/20)+1);
deltas = (1+deltap)/(10^(As/20));
delta = min(deltap,deltas); A = -20*log10(delta);
Deltaw = ws-wp; omegac = (ws+wp)/2;
L = ceil(6.6*pi/Deltaw)+1; M=L-1;
n = [0:M]';
hd =2*omegac/(2*pi)*sinc(2*omegac/(2*pi)*(n-M/2));
h = hd.*hamming(L); stem(n-M/2,h);
w=[0:0.001:1]; H=freqz(h,1,w*pi);
plot(w,20*log10(abs(H)));grid
```


Impulse Response
h(n)
0.3
0.2
0.1
0
−0.1
0
33
66
n
Magnitude Response (dB)
Decibels
0
−50
−100
0
0.2π
0.4π
0.6π
0.8π
π
ω

Adjustable Kaiser Window

$$w[n] = \begin{cases} \frac{I_0[\beta\sqrt{1-[(n-\alpha)/\alpha]^2}]}{I_0(\beta)} & 0 \leq n \leq M, \\ 0 & otherwise \end{cases}$$

$I_0(x) = 1 + \sum_{m=1}^{\infty} (x/2)^m/m!$, $I_0(x)$: 0^{th} order modified Bessel function.

$$\beta = \begin{cases} 0 & A < 1, \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \leq A \leq 50 \\ 0.1102(A-8.7) & A > 50 \end{cases}$$

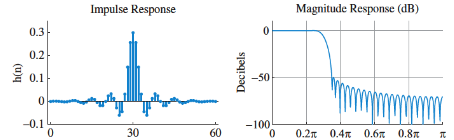
$$\alpha = M/2$$

The order $M = \frac{A-8}{2.285\Delta\omega}$

Example: Design a Kaiser FIR filter with the following specifications: $\omega_p = 0.25\pi$, $\omega_s = 0.35\pi$, $A_p = -0.1$ dB, $A_s = -50$ dB.

- $A = 50$ dB and $\omega_c = (\omega_p + \omega_s)/2 = 0.3\pi$.
- $\beta = 4.528$ and $M = 59$.
- Choose $M = 60$ for a type-I filter.

```
beta =0.5842*(A-21)^(0.4)+0.07886*(A-21);
wp = 0.25*pi; ws = 0.35*pi; Ap = 0.1; As = 50;
Deltaw = ws-wp; omegac = (ws+wp)/2;
M = ceil((A-8)/(2.285*Deltaw))+1; L = M+1;
alpha = M/2; n = 0:M;
hd =2*omegac/(2*pi)*sinc(2*omegac/(2*pi)*(n-M/2));
h = hd.*kaiser(L,beta)';
w=[0:0.001:1]; H=freqz(h,1,w*pi);
plot(w,20*log10(abs(H)));grid
```

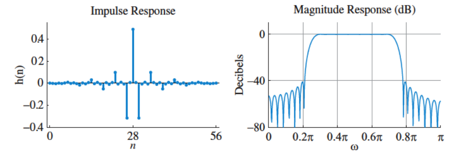


Example: Design a bandpass filter using a Kaiser window

Specifications:

$$
\begin{array}{lll}
 & |H(\omega)| \leq 0.01, & |\omega| \leq 0.2\pi \\
0.99 \leq & |H(\omega)| \leq 1.01 & 0.3\pi \leq |\omega| \leq 0.7\pi, \\
 & |H(\omega)| \leq 0.01, & 0.78 \leq |\omega| \leq \pi
\end{array}
$$

- $h_{bp}[n] = \frac{\sin(\omega_{c2}(n-M/2))}{\pi(n-M/2)} - \frac{\sin(\omega_{c1}(n-M/2))}{\pi(n-M/2)}$
- $\delta_p = 0.01 \rightarrow A \approx 40$ dB,
- $\omega_{c1} = \frac{0.2\pi+0.3\pi}{2} = 0.25\pi$, $\omega_{c2} = \frac{0.7\pi+0.78\pi}{2} = 0.74\pi$,
  $\Delta\omega_1 = 0.3\pi - 0.2\pi = 0.1\pi$, $0.78\pi - 0.7\pi = 0.08\pi$,
- $\beta = 3.395$ and $M = 56$



```
omegac1=.25*pi;
omegac2=0.74*pi;
M=56;
L=57;
n = 0:M;
hd1 =2*omegac1/(2*pi)*sinc(2*omegac1/(2*pi)*(n-M/2));
hd2 =2*omegac2/(2*pi)*sinc(2*omegac2/(2*pi)*(n-M/2));
hh= hd2-hd1;
h = hh.*kaiser(L,beta)';
w = linspace(0,pi,200);
H = freqz(h,1,w);
subplot(2,1,1);stem(h);grid
subplot(2,1,2);plot(w/pi,20*log10(abs(H)));grid
```

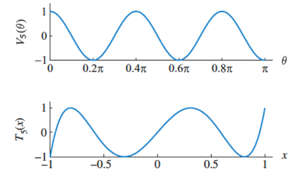
Chebyshev polynomials and minimax approximation

$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x),\ m \geq 1,$
$T_0(x) = 1,\ T_1(x) = x.$

The zeros x_k of the polynomial $T_m(x)$, $|x| < 1$:

$$T_m(x) = \cos(\overbrace{\cos^{-1}(x)}^{\theta}m) = 0 \rightarrow \theta = (2k-1)\pi/(2m)$$

$x_k = \cos\left(\frac{2k-1}{m}\frac{\pi}{2}\right)\ k = 1, 2, \ldots, m.$ $T_m(x)$ has m zeros in $[-1, 1]$.



$\cos(m\theta) = \pm 1$ for $\theta = k\pi/m,\ k = 0, 1, \ldots, m.$
The $m+1$ extrema of $T_m(x)$, $T_m(\xi_k) = (-1)^k$ are $\xi_k = \cos(k\pi/m),\ k = 0, 1, \ldots, m.$

Frequency response function: A trigonometric series in $\cos(\omega m)$ can be expressed $T_m(x)$

$$\begin{aligned} P(\omega) &= 2+\cos(\omega)+\cos(2\omega)+\cos(3\omega) \\ &= [2T_0(x)+T_1(x)+T_2(x)+T_3(x)]|_{x=\cos(\omega)} \\ &= 1-2x+2x^2+4x^3|_{x=\cos(\omega)} \end{aligned}$$

A trigonometric cosine series can be expressed as a polynomial in $\cos(\omega)$

$$P(\omega)=\sum_{k=0}^{R} p[k]\cos(\omega k)=\sum_{k=0}^{R} p[k]T_k(x)|_{x=\cos(\omega)}=\sum_{k=0}^{R}\tilde{p}_k x^k|_{x=\cos(\omega)}$$

Equiripple optimum Chebyshev FIR filter design

$E(\omega) = W(\omega)[A_d(\omega) - A(\omega)]$

Minimax approximation

$$||E(\omega)||_\infty = \max_{\omega \in B} |E(\omega)|$$

$A(\omega) = Q(\omega)P(\omega)$, where

$$Q(\omega) = \begin{cases} 1 & \textit{Type I} \\ \cos(\omega/2) & \textit{Type II} \\ \sin(\omega) & \textit{Type III} \\ \sin(\omega/2) & \textit{Type IV} \end{cases} \quad \text{and } P(\omega) = \sum_{k=0}^{R} p[k]\cos(\omega k)$$

$$E(\omega) = \underbrace{W(\omega)Q(\omega)}_{\bar{W}(\omega)}\Big[\underbrace{\frac{A_d(\omega)}{Q(\omega)}}_{\bar{A}_d(\omega)} - P(\omega)\Big] \qquad R = \begin{cases} \frac{M}{2} & M : \text{even} \\ \frac{M-1}{2} & M : \text{odd} \end{cases}$$

Problem Statement

Given the filter order M, determine the coefficients of $P(\omega)$ that minimize the maximum absolute value of $E(\omega)$ over the frequency bands of interest, that is, choose $P(\omega)$ so that

$$||E(\omega)||_\infty = \max_{\omega \in B} |\bar{W}(\omega)[\bar{A}_d(\omega) - P(\omega)]|$$

is minimum.

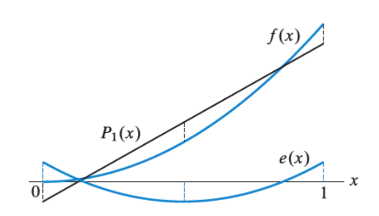
Alternation theorem for FIR filters

A necessary and sufficient condition that $P(\omega)$ be the unique solution of $||E(\omega)||_\infty$ is that the weighted error function $E(\omega)$ exhibit at least $R+2$ alternations in $\mathcal{B}$. That is, there must exist $R+2$ extremal frequencies $\omega_1 < \omega_2 <, \ldots, < \omega_{R+2}$ such that for every $k = 1, 2, \ldots, R+2$,

$$E(\omega_k) = -E(\omega_{k+1}), \quad |E(\omega_k)| = \max_{\omega \in B} E(\omega) = \delta$$

.

Suppose we want to approximate $f(x) = x^2$ i by the polynomial $P_1(x) = x - 0.125$ in the interval $0 \leq x \leq 1$. The error $e(x) = f(x) - P_1(x) = x^2 - x + 0.125$ has $R + 2 = 3$ extrema with the same magnitude and alternating signs: $x(0) = 0.125$, $x(0.5) = -0.125$, and $x(1) = 0.125$.
Hence, the polynomial $P_1(x)$ is the unique minimax approximation to the function $f(x) = x^2$.



Alternation theorem allows for the control of

1. Filter order
2. Edge frequencies of pass bands and stop bands
3. Desired amplitude response $A_d(\omega)$
4. Weighting function for $\omega \in \mathcal{B}$

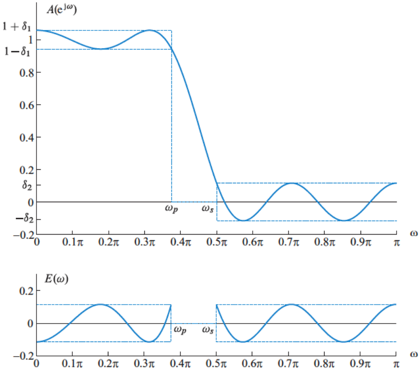
Case Study: Design of M^{th} order type I low pass filter

$$A_d(\omega) = \begin{cases} 1, & 0 \le \omega \le \omega_p \\ 0, & \omega_s \le \omega \le \pi \end{cases} \qquad W(\omega) = \begin{cases} 1, & 0 \le \omega \le \omega_p \\ K = \frac{\delta_p}{\delta_s}, & \omega_s \le \omega \le \pi \end{cases}$$

$$P(\omega) = A(\omega) = \sum_{k=0}^{R} a[k]\cos(\omega k)$$

If $|E(\omega_k)| = \max_{\omega \in B} E(\omega) = \delta$ is not met, M is increased.

If M is even $R = \frac{M}{2}$, $h[0] = h[R]$ and $h[k] = a[R-k]/2$, $k = 1, 2, \ldots R$.



$R = 7$, $K = 1/2$

Minimum number of alternations

$7 + 2 = 9$

Remez Exchange Algorithm

$$E(\omega_i) = W(\omega_i)[A_d(\omega_i) - A(\omega_i)] = (-1)^{i+1}\delta \quad 1 \le i \le R+2$$

$$W(\omega) = \begin{cases} 1, & 0 \le \omega \le \omega_p \\ K = \frac{\delta_p}{\delta_s}, & \omega_s \le \omega \le \pi \end{cases} \qquad P(\omega) = A(\omega) = \sum_{k=0}^{R} a[k]\cos(\omega k)$$

$$\begin{bmatrix} 1 & \cos(\omega_1) & \dots & \cos(R\omega_1) & \frac{1}{W(\omega_1)} \\ 1 & \cos(\omega_2) & \dots & \cos(R\omega_2) & \frac{-1}{W(\omega_2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega_{R+1}) & \dots & \cos(R\omega_{R+1}) & \frac{(-1)^{R+1}}{W(\omega_{R+1})} \\ 1 & \cos(\omega_{R+2}) & \dots & \cos(R\omega_{R+2}) & \frac{(-1)^{R+2}}{W(\omega_{R+2})} \end{bmatrix} \begin{bmatrix} a[0] \\ a[1] \\ \vdots \\ a[R] \\ \delta \end{bmatrix} = \begin{bmatrix} A_d(\omega_1) \\ A_d(\omega_2) \\ \vdots \\ A_d(\omega_{R+1}) \\ A_d(\omega_{R+2}) \end{bmatrix}$$

1. Initialize ω_i which has to include $\omega = 0$, ω_p, ω_s and π and $i = 1, 2, \dots R+1$.
2. Solve the above set of equations for $a[k]$ and δ.
3. Sample $A(\omega) = \sum_{k=0}^{R} a[k]\cos(\omega k)$ to determine the new extremal $\hat{\omega}_i$.
4. Pick up a new set of frequencies $\hat{\omega}_i$, ω_s, ω_p and either 0 or π and repeat steps (2) and (3) until no significant change is observed in $\hat{\omega}_i$.

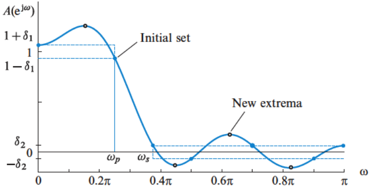
Parks-McClellan Algorithm: A more efficent Approch

1. $$\delta = \frac{\sum_{k=1}^{R+2} b_k A_d(\omega_k)}{\sum_{k=1}^{R+2} \frac{b_k(-1)^{k+1}}{W(\omega_k)}},\ b_k = \prod_{i=1}^{R+2} \frac{1}{x_k - x_i},\ x_i = \cos(\omega_i)$$

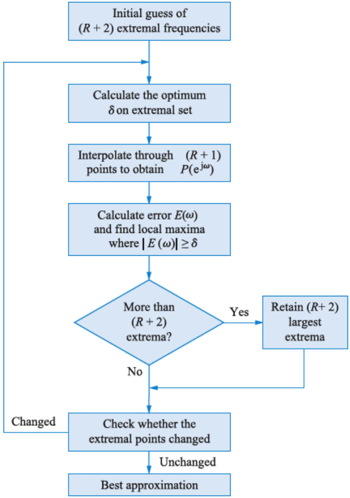
2. $$A(\omega_k) = A_d(\omega_k) - \frac{(-1)^{k+1}\delta}{W(\omega_k)},\ k = 1, 2, \ldots, R+1$$

3. $$A(\omega) = \frac{\sum_{k=1}^{R+1} A(\omega_k)\frac{d_k}{x - x_k}}{\sum_{k=1}^{R+1} \frac{d_k}{(x - x_k)}} \quad d_k = \prod_{i=0}^{R+1} \frac{1}{(x_k - x_i)} = b_k(x_k - x_{R+2})$$

$$x = \cos(\omega)$$



$A(e^{j\omega})$
$1+\delta_1$
1
$1-\delta_1$
Initial set
New extrema
δ_2
0
$-\delta_2$
ω_p
ω_s
0
0.2π
0.4π
0.6π
0.8π
π
ω



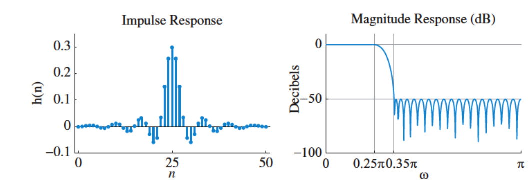
Initial guess of $(R+2)$ extremal frequencies
Calculate the optimum δ on extremal set
Interpolate through $(R+1)$ points to obtain $P(e^{j\omega})$
Calculate error $E(\omega)$ and find local maxima where $|E(\omega)| \geq \delta$
More than $(R+2)$ extrema?
Yes
Retain $(R+2)$ largest extrema
No
Changed
Check whether the extremal points changed
Unchanged
Best approximation

Example: Design a low-pass filter with specifications: $\omega_p = 0.25\pi$, $\omega_s = 0.35\pi$, $A_p = 0.1\text{dB}$, $A_s = 50\text{dB}$

$\delta_p = 0.0058$, $\delta_s = 0.0032$

```
wp = 0.25*pi; ws = 0.35*pi; Ap = 0.1; As = 50;
deltap =(10^(Ap/20)-1)/(10^(Ap/20)+1);
deltas = (1+deltap)/(10^(As/20));
[M,fo,ao,W] = firpmord([wp,ws]/pi,[1,0],[deltap,deltas]);
[h,delta] = firpm(M,fo,ao,W);
```

$M = 48$ and $\delta = 0.0071$
$M = 50$ and $\delta = 0.0055$



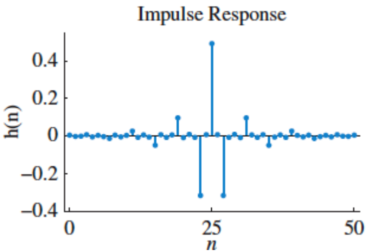
Example: Design a bandpass filter

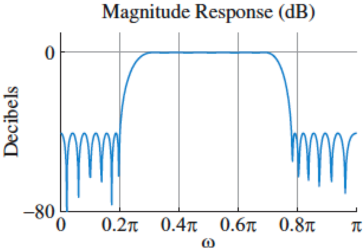
Specifications:

$$\begin{array}{rll} & |H(\omega)| \le 0.01, & |\omega| \le 0.2\pi \\ 0.99 \le & |H(\omega)| \le 1.01 & 0.3\pi \le |\omega| \le 0.7\pi, \\ & |H(\omega)| \le 0.01, & 0.78 \le |\omega| \le \pi \end{array}$$

- $h_{bp}[n] = \frac{\sin(\omega_{c2}(n-M/2))}{\pi(n-M/2)} - \frac{\sin(\omega_{c1}(n-M/2))}{\pi(n-M/2)}$
- $\delta_p = 0.01 \rightarrow A \approx 40$ dB,
- $\omega_{c1} = \frac{0.2\pi+0.3\pi}{2} = 0.25\pi$, $\omega_{c2} = \frac{0.7\pi+0.78\pi}{2} = 0.74\pi$,
- $\Delta\omega_1 = 0.3\pi - 0.2\pi = 0.1\pi$,

```
ws1 = 0.2*pi; deltas1 = 0.01;
wp1 = 0.3*pi; wp2 = 0.7*pi; deltap = 0.01;
ws2 = 0.78*pi; deltas2 = 0.01;
f = [ws1,wp1,wp2,ws2]/pi; a = [0,1,0];
dev = [deltas1,deltap,deltas2];
[M,fo,ao,W] = firpmord(f,a,dev);
[h,delta] = firpm(M,fo,ao,W);
```


Impulse Response
h(n)
0.4
0.2
0
–0.2
–0.4
0
25
50
n


Magnitude Response (dB)
Decibels
0
–80
0
0.2π
0.4π
0.6π
0.8π
π
ω