Fourier Series Representation of Discrete Periodic Signals Lecture Notes

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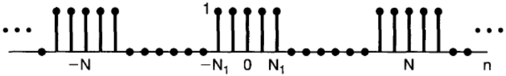
Some concepts and illustrations in this lecture are adapted from the textbook,
Signals and Systems, 2nd Edition by Alan Oppenheim, Alan Willisky and H. Nawab, *Prentice Hall*.

Discrete Time Fourier Series Representation

Express $x[n] = x[n+N]$ as a linear sum of complex sinusoids $\phi_k[n] = e^{jk\omega_0 n}$.
There are at most N distinct ϕ_k i.e. $\phi_0, \phi_1, \ldots, \phi_{N-1}$ since $\phi_k[n] = \phi_{k+rN}[n]$, r being and integer.

$$x[n] = \sum_{k=<N>} a_k e^{j\omega_0 nk} \longleftrightarrow a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j\omega_0 nk}$$

Example



$$a_k = \frac{1}{N}\frac{\sin(2\pi k(N_1 + 1/2)/N)}{\sin(\pi k/N)}, \qquad k \neq 0, \pm N, \pm 2N, \ldots$$

$$a_k = \frac{2N_1 + 1}{N}, \qquad k = 0, \pm N, \pm 2N, \ldots$$

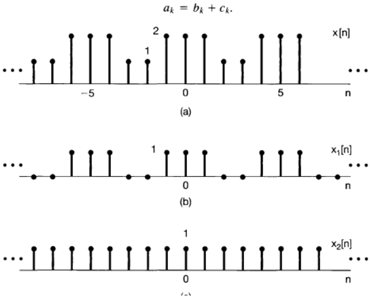
Properties of DTFS

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

Examples



$$b_k = \begin{cases} \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)} & \textit{for } k \neq 0, \pm 5, \pm 10, \dots \\ \frac{3}{5} & \textit{for } k = 0, \pm 5, \pm 10, \dots \end{cases}$$

$$c_0 = 1, \; c_k \neq 0$$

$$a_k = b_k + c_k$$

Algebraic Formulation of Discrete Fourier Transform (DFT)

$$x[n] = \sum_{k=<N>} a_k e^{j\omega_0 nk} \longleftrightarrow X[k] = \sum_{n=<N>} x[n] e^{-j\omega_0 nk}$$

Defining $W_N = e^{-j\frac{2\pi}{N}}$

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & W_N & \dots & W_N^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

In short notation,

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N \longleftrightarrow \mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N$$
$$\mathbf{W}_N^{-1} = \frac{1}{N} \mathbf{W}_N^*$$

N point DFTs of Two Real Sequences, $g[n]$ and $h[n]$

$x[n] = g[n] + jh[n]$

$$G[k] = \frac{1}{2}\{X[k] + X^*[N-k]\}$$

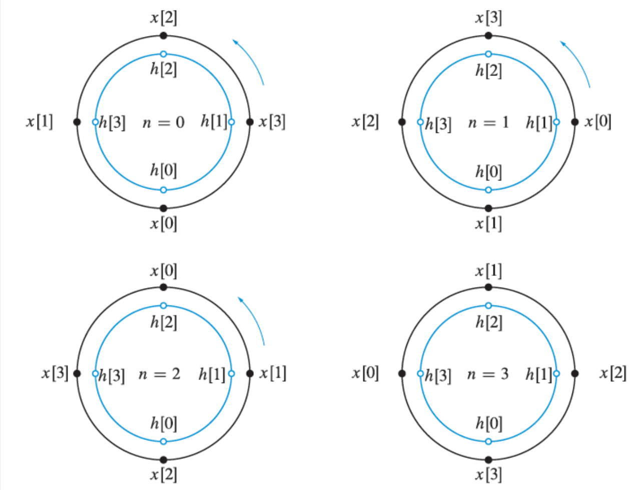
$$H[k] = \frac{1}{2j}\{X[k] - X^*[N-k]\}$$

$2N$ point DFT of a Real Sequence $v[n]$

$g[n] = v[2n]$ and $h[n] = v[2n+1]$, $0 \leq n < N$. $V[k] = \sum_{n=0}^{2N-1} v[n]e^{-j\frac{2\pi}{2N}kn} = \sum_{n=0}^{N-1} v[2n]e^{-j\frac{2\pi}{2N}k2n} + \sum_{n=0}^{N-1} v[2n+1]e^{-j\frac{2\pi}{2N}k(2n+1)}$
$\sum_{n=0}^{N-1} g[n]e^{-j\frac{2\pi}{N}kn} + e^{-j\frac{2\pi}{2N}k} \sum_{n=0}^{N-1} h[n]e^{-j\frac{2\pi}{N}kn}$

$$V[k] = G[k] + e^{-j\frac{2\pi}{2N}k}H[k], \quad 0 \leq k \leq 2N-1$$

Circular Convolution of $x[n]$ and $h[n]$



Some Further Properties of Discrete Convolution and DFT

Circular Convolution of sequences $x[n]$ and $y[n]$ of length N

$$x[n] \circ y[n] \leftrightarrow N \text{ point } IDFT\{DFT\{x[n]\}DFT\{y[n]\}\}$$

Linear Convolution of sequences $x[n]$ of length N and $y[n]$ of M

$$x[n] * y[n] \leftrightarrow N + M - 1 \text{ point } IDFT\{DFT\{x[n]\}DFT\{y[n]\}\}$$

Convolution of an N point $x[n]$ with M point $h[n]$ for $N >> M$

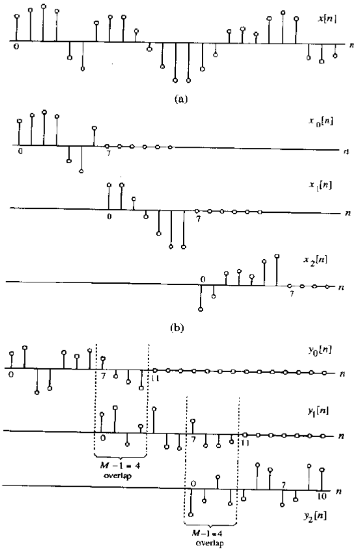
Overlap Add

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \\ y[7] \\ y[8] \\ y[9] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 & 0 & 0 & 0 & 0 \\ h[2] & h[1] & h[0] & 0 & 0 & 0 & 0 & 0 \\ 0 & h[2] & h[1] & h[0] & 0 & 0 & 0 & 0 \\ 0 & 0 & h[2] & h[1] & h[0] & 0 & 0 & 0 \\ 0 & 0 & 0 & h[2] & h[1] & h[0] & 0 & 0 \\ 0 & 0 & 0 & 0 & h[2] & h[1] & h[0] & 0 \\ 0 & 0 & 0 & 0 & 0 & h[2] & h[1] & h[0] \\ 0 & 0 & 0 & 0 & 0 & 0 & h[2] & h[1] \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & h[2] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}.$$

Overlap Save

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \\ y[7] \\ y[8] \\ y[9] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 & 0 & 0 & 0 & 0 \\ h[2] & h[1] & h[0] & 0 & 0 & 0 & 0 & 0 \\ 0 & h[2] & h[1] & h[0] & 0 & 0 & 0 & 0 \\ 0 & 0 & h[2] & h[1] & h[0] & 0 & 0 & 0 \\ 0 & 0 & 0 & h[2] & h[1] & h[0] & 0 & 0 \\ 0 & 0 & 0 & 0 & h[2] & h[1] & h[0] & 0 \\ 0 & 0 & 0 & 0 & 0 & h[2] & h[1] & h[0] \\ 0 & 0 & 0 & 0 & 0 & 0 & h[2] & h[1] \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & h[2] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}.$$

Overlap Add



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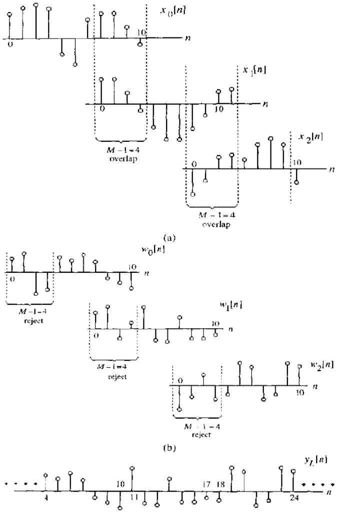

	Property	N-point sequence	N-point DFT
		$x[n], h[n], v[n]$	$X[k], H[k], V[k]$
		$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
1.	Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1[k] + a_2X_2[k]$
2.	Time shifting	$x[\langle n-m\rangle_N]$	$W_N^{km}X[k]$
3.	Frequency shifting	$W_N^{-mn}x[n]$	$X[\langle k-m\rangle_N]$
4.	Modulation	$x[n]\cos(2\pi/N)k_0n$	$\frac{1}{2}X[\langle k+k_0\rangle_N] + \frac{1}{2}X[\langle k-k_0\rangle_N]$
5.	Folding	$x[\langle -n\rangle_N]$	$X^*[k]$
6.	Conjugation	$x^*[n]$	$X^*[\langle -k\rangle_N]$
7.	Duality	$X[n]$	$Nx[\langle -k\rangle]_N]$
8.	Convolution	$h[n] \textcircled{N} x[n]$	$H[k]X[k]$
9.	Correlation	$x[n] \textcircled{N} y[\langle -n\rangle_N]$	$X[k]Y^*[k]$
10.	Windowing	$v[n]x[n]$	$\frac{1}{N}V[k] \textcircled{N} X[k]$
11.	Parseval's theorem	$\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]Y^*[n]$	
12.	Parseval's relation	$\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$	