Discrete Linear Systems
Lecture Notes

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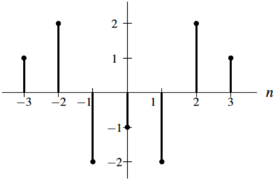
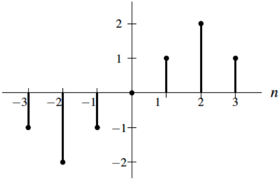
Properties of Discrete Signals

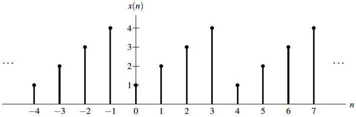
Even sequence $x[n]$



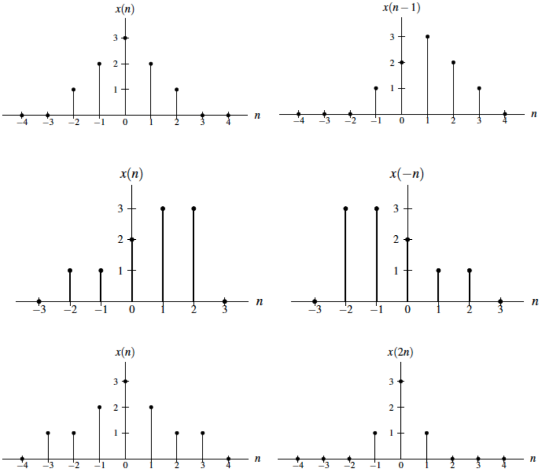
Odd sequence $x[n]$



Periodic sequence with period N: $x[n] = x[n + N]$



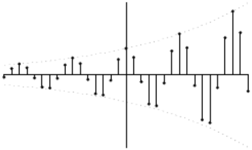
Shifting, Reversing and Scaling of Discrete Signals



Exponential and Sinusoidal Signals

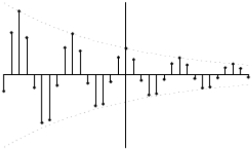
$$x[n] = |C|e^{j\theta}e^{(a+j\omega_0)n}$$

Real part

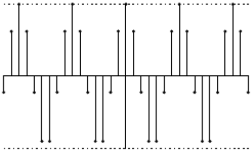


$|a| > 1$

Imaginary Part

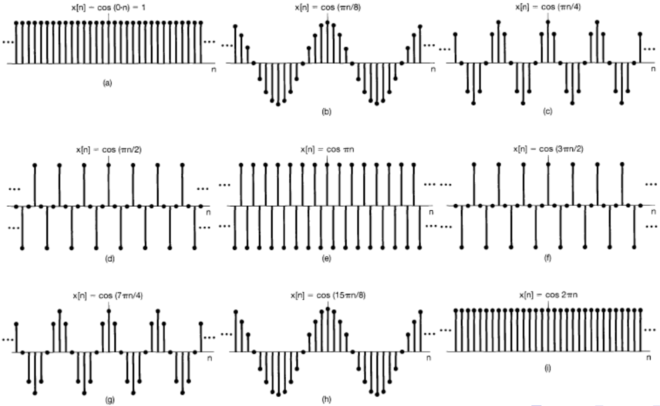


$|a| < 1$



Periodicity of Discrete Sinusoids

We can have at most N harmonics of a sinusoid with fundamental frequency $\omega_0 = \frac{2\pi}{N}$ ranging from $0, \omega_0, 2\omega_0, \ldots, (N-1)\omega_0$.



Example

Example:
What is the fundamental period of $x[n]$

$$x[n] = e^{j2\pi/3n} + e^{j2\pi/4n}$$

Example:
Is the system whose input $x[n]$ and output $y[n]$ are given as $y[n] = x[4n+1]$ causal, stable, linear, time-invariant?

Example:
Sketch $-x[-2n+1]$ if $x = \{1, 2, \hat{3}, 0, 0, 0, 0, 0, 1\}$ where ^symbol denotes $n = 0$.

Unit step and unit Impulse Sequences

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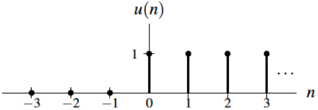
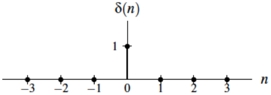




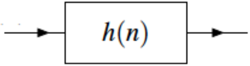
$$u[n] = \sum_{k=-\infty}^{n} \delta[k] \qquad \delta[n] = u[n] - u[n-1]$$

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0] \qquad \sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]$$

Energy of a discrete signal $x[n]$ is $E = \sum_{n=-\infty}^{\infty} |x[n]|^2$

Impulse Response of LTI Discrete Systems

$\delta[n] \longleftrightarrow y[n] = h[n]$



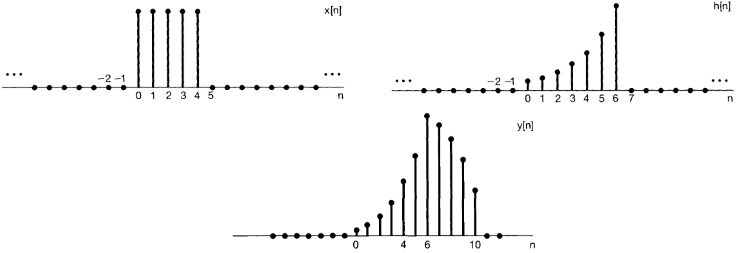
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * y[n]$$

If the system is causal then the impulse response $h[n] = 0$ for $n < 0$ and the output $y[n] = \sum_{k=0}^{\infty} x[n-k]h[k] = \sum_{k=0}^{\infty} x[k]h[n-k]$

Example

$$x[n] = \begin{cases} 1 & 0 \le n \le 4, \\ 0 & otherwise \end{cases} \quad , \quad h[n] = \begin{cases} \alpha^n & 0 \le n \le 6, \\ 0 & otherwise \end{cases}$$



Discrete Convolution as a Matrix Notation

$$\mathbf{y} = \begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix} \mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[M-1] \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} h[0] & 0 & \dots & 0 \\ h[1] & h[0] & \dots & 0 \\ h[2] & h[1] & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ h[p-1] & h[p-2] & \dots & h[p-M] \end{bmatrix}$$

The output is $\mathbf{y} = \mathbf{H}\mathbf{x}$

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x = ones(5,1); a=1.5; h =a.^[0:6]'; H = convmtx(h,5);
y = H*x; stem(y);
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LTI Systems Described by Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

Example:
Determine the output of the system given by

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

to an input $x[n] = K\delta[n]$.
$y[n] = \left(\frac{1}{2}\right)^n K$

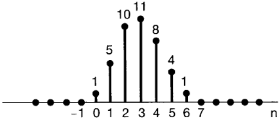
Example

$$y[n] = \frac{1}{4}y[n-1] + x[n]$$

Determine $y[n]$ if $x[n] = \delta[n-1]$.

$$y[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]$$

Consider the cascade interconnection of three causal LTI systems



The impulse response $h_2[n]$ is $h_2[n] = u[n] - u[n-2]$, and the overall impulse response is

1. Find the impulse response $h_1[n]$.
2. Find the response of the overall system to the input $x[n] = \delta[n] - \delta[n-1]$.

$h[n] = h_1[n] + 2h_1[n-1] + h_1[n-2]$