Discrete Time Fourier Transform (DTFT) Lecture Notes

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Some concepts and illustrations in this lecture are adapted from the textbook,
Signals and Systems, 2nd Edition by Alan Oppenheim, Alan Willisky and H. Nawab, *Prentice Hall*.

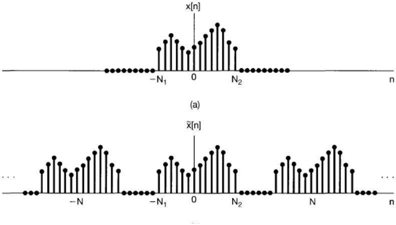
Discrete Time Fourier Series Representation

Express $x[n] = x[n+N]$ as a linear sum of complex sinusoids $\phi_k[n] = e^{jk\omega_0 n}$.

There are at most N distinct ϕ_k i.e. $\phi_0, \phi_1, \ldots, \phi_{N-1}$ since $\phi_k[n] = \phi_{k+rN}[n]$, r being and integer.

$$x[n] = \sum_{k=<N>} a_k e^{j\omega_0 nk} \longleftrightarrow a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j\omega_0 nk}$$

Fourier Series Representation of Non Periodic Sequences

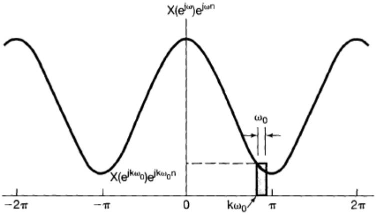


$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \longrightarrow a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n} = \frac{X(k\omega_0)}{N} \longrightarrow X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Fourier Series Representation of Non Periodic Sequences

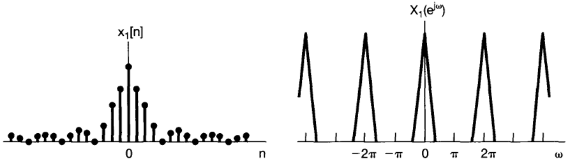
$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X(k\omega_0) e^{jk\omega_0 n} = \sum_{k=<N>} \frac{1}{2\pi} X(k\omega_0) e^{jk\omega_0 t} \omega_0$$



$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

Discrete Time Fourier Transform Pair

DTFT : Special Case of $z-$Transform where $z = e^{j\omega}$

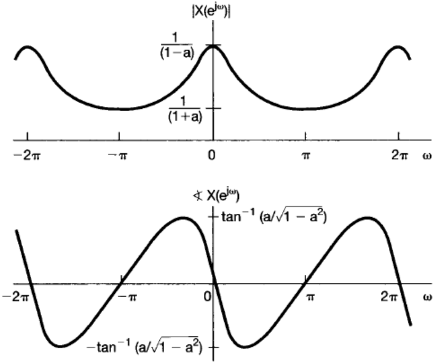


$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Example

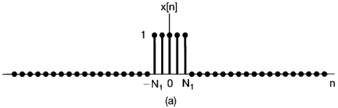
$$x[n] = a^n u[n] \qquad |a| < 1, \longleftrightarrow \frac{1}{1 - ae^{-j\omega}}$$

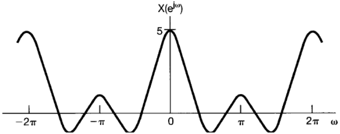


Examples

$$x[n] = a^{|n|} \quad |a| < 1, \longleftrightarrow \frac{1-a^2}{1-2a\cos(\omega)+a^2}$$

$$x[n] = \begin{cases} 1 & |n| \leq N_1, \\ 0 & otherwise \end{cases} \longleftrightarrow X(\omega) = \frac{\sin(\omega(N1+\frac{1}{2}))}{\sin(\omega/2)}$$





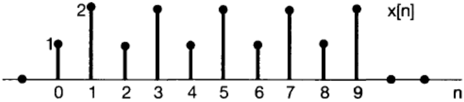
Properties of DTFT

- Periodicity : $X(\omega + 2\pi) = X(\omega)$
- *Linearity* : $ax_1[n] + bx_2[n] \longleftrightarrow aX_1(\omega) + bX_2[\omega]$
- Time Shifting : $x[n - n_0] \longleftrightarrow e^{-j\omega n_0} X(\omega)$
- Frequency Shifting : $e^{j\omega_0 n} \longleftrightarrow X(\omega - \omega_0))$
- Conjugation : $x^*[n] \longleftrightarrow X^*(-\omega)$
  If $x[n]$ is real
- $X(\omega) = X^*(-\omega)$,
  $\mathcal{E}v\{x[n]\} \longleftrightarrow \mathcal{R}e\{X(\omega)\}$, $\mathcal{O}dd\{x[n]\} \longleftrightarrow j\mathcal{I}m\{X(\omega)\}$
- Accumulation:
  $$y[n] = \sum_{m=-\infty}^{n} x[m] \longleftrightarrow \frac{1}{1-e^{-j\omega}} X(\omega) + \pi X(1) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Properties of DTFT

- Time Reversal: $x[-n] \longleftrightarrow X(-\omega)$
- Time Expansion :
$$x_{(k)}[n] = \begin{cases} x(n/k), & \textit{if } n \textit{ is multiple of } k \\ 0 & \textit{otherwise} \end{cases} \longleftrightarrow X(k\omega)$$
- Differentiation in Frequency : $nx[n] \longleftrightarrow j\frac{d}{d\omega}X(\omega)$
- Parseval's Relation : $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\omega)|^2 d\omega$
- Convolution : $x[n] * y[n] \longleftrightarrow X(\omega)Y(\omega)$
- Multiplication : $x[n]y[n] \longleftrightarrow \frac{1}{2\pi} \int_{2\pi} X(\theta)Y(\omega - \theta)d\theta$

Examples

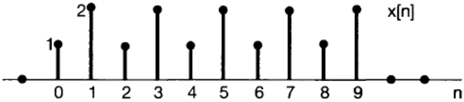


$$h_{hp}(\omega) = h_{lp}(\omega - \pi) \longleftrightarrow h_{hp}[n] = e^{j\pi n} h_{lp}[n] = (-1)^n h_{lp}[n]$$

Examples

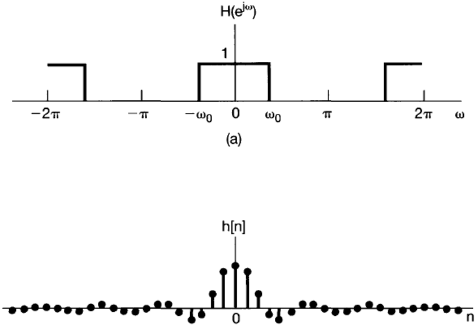
$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

$$U(\omega) = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$



$$x[t] = y_{(2)}[n] + 2y_{(2)}[n-1] \longleftrightarrow X(\omega) = e^{-j4\omega}(1+2e^{-j\omega})\left(\frac{\sin(5\omega)}{\sin(\omega)}\right)$$

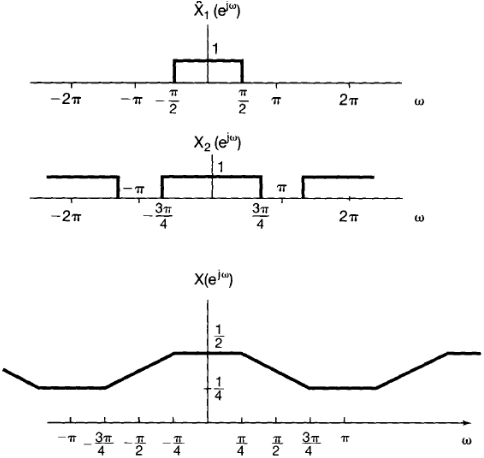
Example



$$H(\omega) \longleftrightarrow \frac{\sin(\omega_c n)}{\pi n}$$

Example

$$x[n] = x_1[n]x_2[n] \qquad \text{where } x_1[n] = \frac{\sin(3\pi n/4)}{\pi n},\ x_2[n] = \frac{\sin(\pi n/2)}{\pi n}$$



Example

$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$ with $|a| < 1$.
Determine $y[n]$ for $x[n] = \left(\frac{1}{4}\right)^n u[n]$.

$$Y(\omega) = -\frac{4}{1 - \frac{1}{4}e^{-j\omega}} - \frac{2}{\left(1 - \frac{1}{4}e^{-j\omega}\right)^2} + \frac{8}{1 - \frac{1}{2}e^{-j\omega}}$$

$$y[n] = \left\{ -4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n \right\} u[n]$$

Example

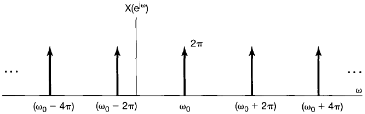
Determine the output of the LTI system with impulse response $h[n] = \alpha^n u[n]$, $-1 < \alpha < 1$, to the input $x[n] = \cos\left(\frac{2\pi n}{N}\right)$
$H(\omega) = \frac{1}{1-\alpha e^{-j\omega}}$

$$e^{j\omega n} \longrightarrow H(\omega)e^{j\omega n}$$

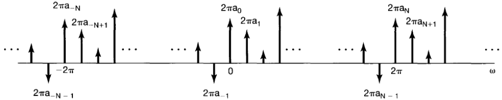
$$y[n] = \frac{1}{2}H(2\pi/N)e^{j(2\pi/N)n} + \frac{1}{2}H(-2\pi/N)e^{-j(2\pi/N)n}$$

DTFT for Periodic Sequences

$$e^{j\omega_0 n} = \frac{1}{2\pi}\int_{2\pi} 2\pi\delta(\omega - \omega_0 - 2\pi l)e^{j\omega n}dw$$

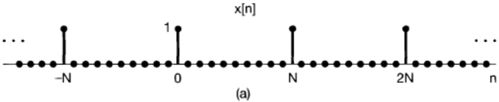


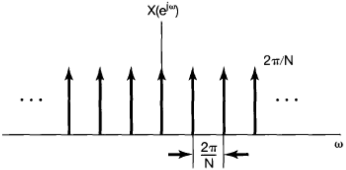
$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \longleftrightarrow X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})$$



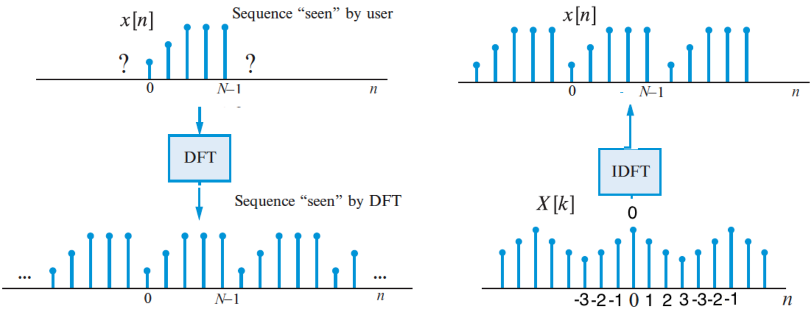
Example

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \qquad X(\omega) = \frac{1}{N} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \frac{2\pi k}{N})$$

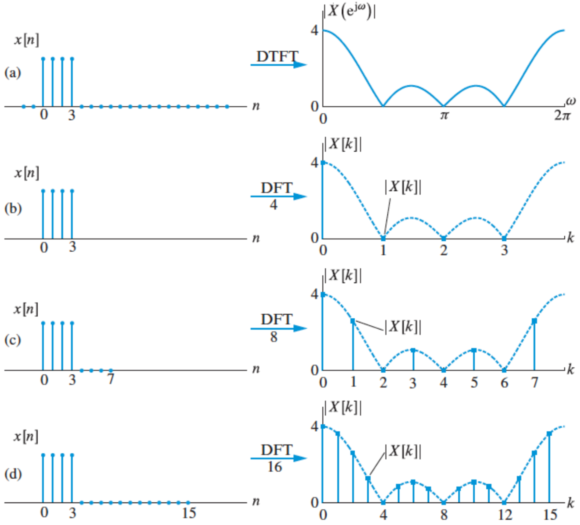




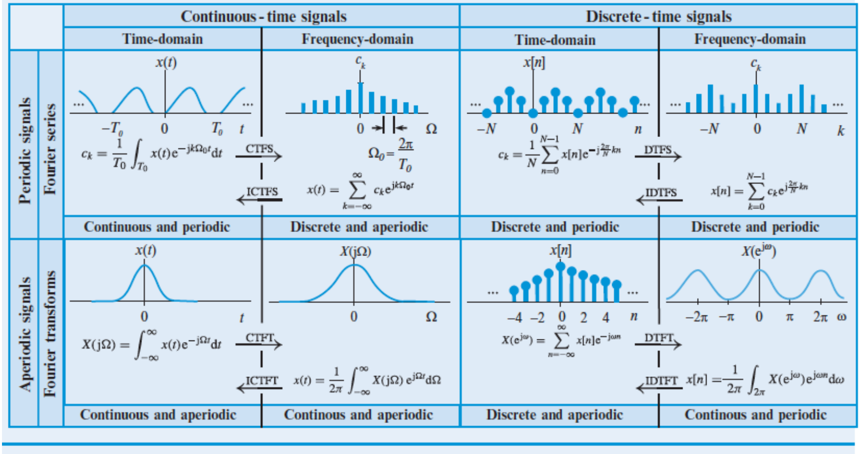
The Relation Between DTFT and DFT



Effect of Zero Padding



Summary of Fourier Representations



Reconstructing Discrete Time Fourier Transform from DFT

$$X(\omega) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = X(\omega) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\omega_0 nk}\right]e^{-j\omega n}, \text{ with } \omega_0 = \frac{2\pi}{N}.$$

$$X(\omega) = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\left[\sum_{n=0}^{N-1} e^{-j(\omega-\omega_0 k)n}\right] = \sum_{k=0}^{N-1} X[k]\frac{\sin[\frac{N}{2}(\omega-\omega_0 k)]}{N\sin[\frac{1}{2}(\omega-\omega_0 k)]}e^{-j(\omega-\omega_0 k)(\frac{N-1}{2})}$$

A sample MATLAB code to obtain DTFT from DFT

```
clear i;
N =11; M=100; % define the data lenght for discrete signal and its DTFT
x = 10*rand(N,1); % generate a signal
X = fft(x); % Compute its DFT
w0= 2*pi/N; % Determine frequency spacing in radians
k=0:N-1; % Define DFT indices
w = linspace(-pi,pi,M)'; % Define DTFT intervals
% Compute DTFT from DFT coefficients and plot the DTFT and DFT
f = w*ones(size(k)) -  ones(size(w))*k*w0;
P = sin(N/2*f)./sin(1/2*f).*exp(-i*(N-1)/2*f); Xw = P*X/N;
subplot(2,1,1);plot(abs(Xw)); subplot(2,1,2);stem(fftshift(abs(X)));
```