Continuous Time Fourier Transform
Lecture Notes

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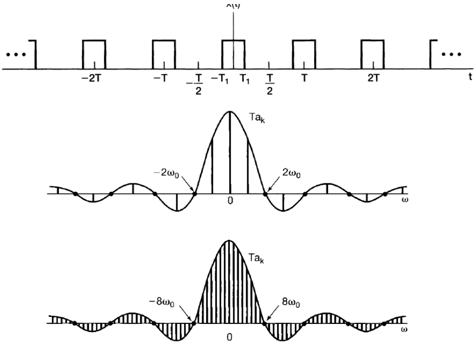
Some concepts and illustrations in this lecture are adapted from the textbook,
Signals and Systems, 2nd Edition by Alan Oppenheim, Alan Willisky and H. Nawab, *Prentice Hall*.

Fourier Series of Continuous Nonperiodic Signals

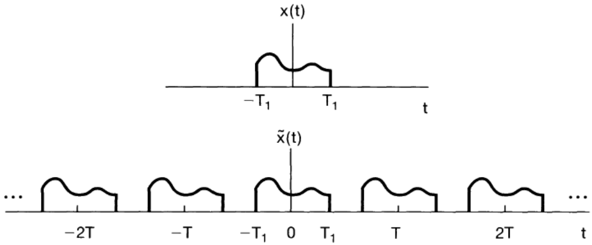


$$x_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$$T = 4T_1$$

$$T = 16T_1$$

Fourier Series of Continuous Nonperiodic Signals

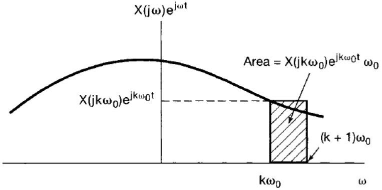


$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt = \frac{X(k\omega_0)}{T} \longrightarrow X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Series of Continuous Nonperiodic Signals

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(k\omega_0) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} X(k\omega_0) e^{jk\omega_0 t} \omega_0$$



when $T \to \infty$

$\omega_0 \to d\omega$

$\tilde{x}(t) \to x(t)$

Fourier Transform and its Inverse for continuous nonperiodic signal

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \longleftrightarrow X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Continuous Fourier Transform and its Inverse Pair

Special case of Laplace Transform $X(s) = \int\limits_{-\infty}^{\infty} x(t)e^{-st}dt$

where $s = j\omega$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

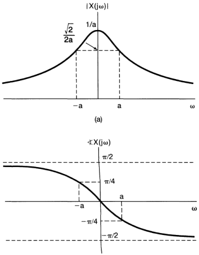
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

Example

- $x(t) = e^{-at}u(t) \qquad a > 0,$

$$X(\omega) = \frac{1}{a + j\omega}$$



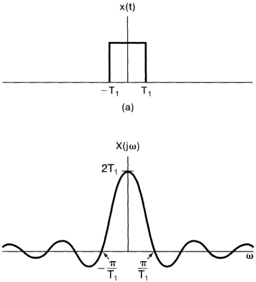
- $x(t) = e^{-a|t|}u(t) \qquad a > 0,$

$$X(\omega) = \frac{2a}{a^2 + \omega^2}$$

Example

$$x(t) = \begin{cases} 1 & |t| < T_1, \\ 0 & |t| > T_1 \end{cases}$$

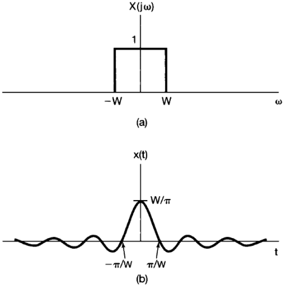
$$X(\omega) = 2\frac{sin(\omega T_1)}{\omega}$$



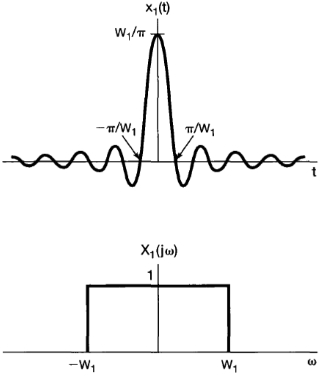
Example

$$X(\omega) = \begin{cases} 1 & |\omega| < W, \\ 0 & |\omega| > W \end{cases}$$

$$x(t) = \frac{sin(Wt)}{\pi t}$$



Inverse Relation between the Time and Frequency Domains



Fourier Transform of Periodic Signals

$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

Example:

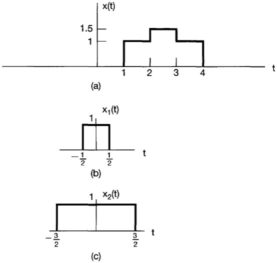
$$x(t) = \sin(\omega_0 t)$$

$$X(\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

Properties of Continuous Fourier Transform

- Linearity : $ax(t) + by(t) \overset{\mathcal{F}}{\longleftrightarrow} aX(\omega) + bY(\omega)$
- Time Shifting: $x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0}X(\omega)$

Example: $x(t) = \frac{1}{2}x_1(t - 2.5) - x2(t - 2.5)$



$$X(\omega) = e^{-j5\omega/2}\left\{\frac{sin(\omega/2+2sin(3\omega/2))}{\omega}\right\}$$

Properties of Continuous Fourier Transform

- Conjugation and Conjugate Symmetry
  $x^*(t) \longleftrightarrow X^*(-\omega)$
  $x(-t) \longleftrightarrow X(-\omega)$
  If $x(t)$ is real, $X(-\omega) = X^*(\omega)$
  If $x(t)$ is even, $X(-\omega) = X(\omega)$
  If $x(t)$ is odd, $X(-\omega) = -X(\omega)$
  $\text{Even}\{x(t)\} \stackrel{\mathcal{F}}{\longleftrightarrow} \text{Re}\{X(\omega)\}$
  $\text{Odd}\{x(t)\} \stackrel{\mathcal{F}}{\longleftrightarrow} j\text{Im}\{X(\omega)\}$
  Example:
  $x(t) = e^{-a|t|} = e^{at}u(-t) + e^{-at}u(t) = 2\frac{e^{at}u(-t)+e^{-at}u(t)}{2}$
  $X(\omega) = 2Re\{\frac{1}{a+j\omega}\} = \frac{2a}{a^2+\omega^2}$

Properties of Continuous Fourier Transform

- Convolution: $x(t) * y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)Y(\omega)$
- Differentiation: $\frac{dx(t)}{dt} \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega X(\omega)$
- Integration : $\int\limits_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$

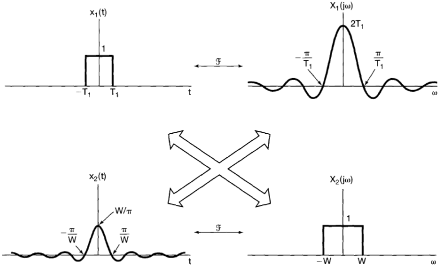
$$\int\limits_{-\infty}^{t} x(\tau)d\tau = \int\limits_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)U(\omega)$$

$$u(t) = \frac{1}{2}(sgn(t) + 1) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}(\frac{2}{j\omega} + 2\pi\delta(\omega))$$

$$\int\limits_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)(\frac{1}{j\omega} + \pi\delta(\omega))$$

- Time and Frequency Scaling : $x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X(\frac{\omega}{a})$

Properties of Continuous Fourier Transform



Example: If $e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{1+\omega^2}$ then $\frac{2}{1+t^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi e^{|\omega|}$

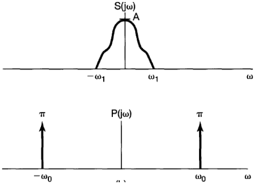
$$e^{-|t|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+\omega^2} e^{j\omega t} d\omega \longleftrightarrow 2\pi e^{-|t|} = \int_{-\infty}^{\infty} \frac{2}{1+\omega^2} e^{j\omega t} d\omega$$

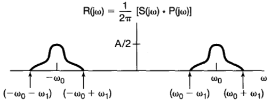
$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty} \frac{2}{1+t^2} e^{-j\omega t} dt$$

Properties of Continuous Fourier Transform

- Differentiation in Frequency: $-jtx(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{dX(\omega)}{d\omega}$
- Shift in Frequency: $e^{j\omega_0 t}x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega - \omega_0)$
- Parseval's Relation: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$
- $z(t) = x(t)y(t) \overset{\mathcal{F}}{\longleftrightarrow} Z(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\theta)Y(\omega - \theta)d\theta$

Example: $p(t) = \cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} P(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega - \omega_0)$





Systems Characterized by Linear Differential Equations

Example:

1. $\frac{dy(t)}{dt} + ay(t) = x(t)$ with $a > 0$

$$H(\omega) = \frac{1}{a + j\omega}$$

$$h(t) = e^{-at}u(t)$$

2. $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$ with $a > 0$

$$x(t) = e^{-at}u(t), X(\omega) = \frac{1}{a + j\omega}$$

$$Y(\omega) = \frac{1/4}{1 + j\omega} + \frac{1/2}{(1 + j\omega)^2} - \frac{1/4}{3 + j\omega}$$

$$y(t) = (\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t})u(t)$$