Introduction to Linear Systems
Lecture Notes

Ahmet Ademoglu, *PhD*
Bogazici University
Institute of Biomedical Engineering

Some concepts and illustrations in this lecture are adapted from the textbook,
Signals and Systems, 2nd Edition by Alan Oppenheim, Alan Willisky and H. Nawab, *Prentice Hall*.

Attributes of Signals/Systems

- Linear–Nonlinear
- Time (Shift) Invariant–Variant
- Discrete–Continuous
- Causal–Noncausal
- Static–Dynamic
- Stable–Unstable
- Deterministic–Random
- Invertible
- Energy–Power

Linearity

- Additivity:
  $x_1(t) \longrightarrow y1(t)$
  $x_2(t) \longrightarrow y2(t)$
  $x_1(t) + x_2(t) \longrightarrow y1(t) + y_2(t)$
- Homogeneity
  $x(t) \longrightarrow y(t)$
  $\alpha x(t) \longrightarrow \alpha y(t)$

Time-invariance (Shift-invariance)

- $x(t) \rightarrow y(t)$
  $x(t-\tau) \rightarrow y(t-\tau)$

Important Signals

- Impulse function $\delta(t)$
- Step function $u(t)$
- Ramp function $r(t)$
- Exponential sinusoid $e^{\sigma t} e^{j\omega t}$

Causality

- The current output of the system $y(t)$ is only a function of the current and past inputs,
  $y(t) = f(x(t - \tau))\, \tau \geq 0$.

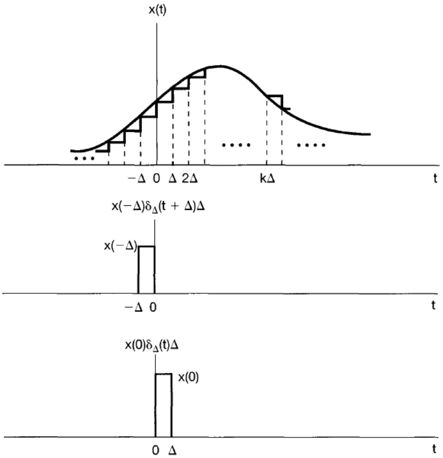
More Examples :

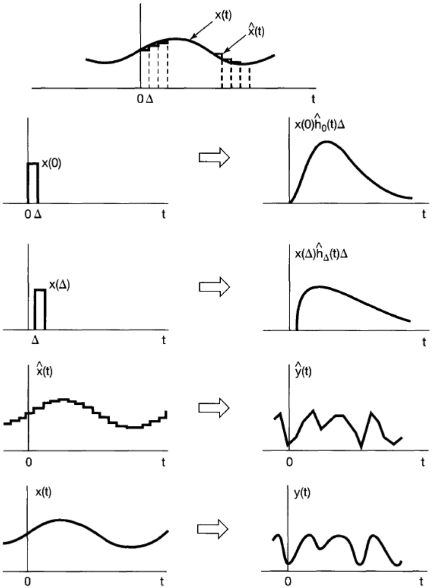
Show if the systems below are i) linear ii) time invariant,

1. $y(t) = t\frac{dx(t)}{dt} + x(t)$
2. $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$
3. $y(t) = \begin{cases} 1 & x(t) < 0, \\ x(t) + x(t-2) & otherwise \end{cases}$

Convolution Integral : $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$




x(t)
x̂(t)
0 Δ
t
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x(0)ĥ0(t)Δ
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x(Δ)ĥΔ(t)Δ
Δ
x̂(t)
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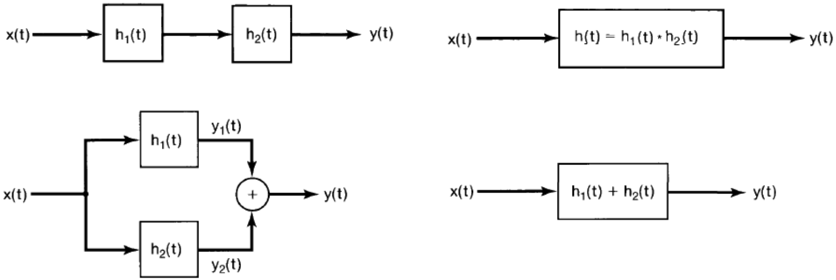
x(t)
y(t)

Interconnection of the Systems



Cascade LTI systems

$$w(t) = \int_{-\infty}^{\infty} x(t-\tau)h_1(\tau)d\tau \text{ and } y(t) = \int_{-\infty}^{\infty} w(t-\alpha)h_2(\alpha)d\alpha$$

$$y(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\alpha-\tau)h_1(\tau)h_2(\alpha)d\tau d\alpha$$

$$y(t) = \int_{-\infty}^{\infty} x(t-\beta)\underbrace{\int_{-\infty}^{\infty} h_1(\tau)h_2(\beta-\tau)d\tau}_{h(\beta)} d\beta$$

$$y(t) = \int_{-\infty}^{\infty} x(t-\beta)h(\beta)d\beta$$

Examples: Determine $y(t) = x(t) * h(t)$

$$x(t) = e^{-at}u(t), \quad a > 0 \qquad \text{and } h(t) = u(t)$$

$$x(t) = \begin{cases} 1 & 0 < t < T, \\ 0 & \textit{otherwise} \end{cases}$$

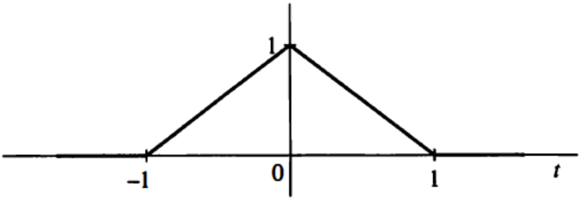
$$h(t) = \begin{cases} t & 0 < t < 2T, \\ 0 & \textit{otherwise} \end{cases}$$

More Examples : Evaluate the following integrals

1. $\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau,$
2. $\int_{-\infty}^{\infty}(t^3+4)\delta(1-t)dt,$
3. $\int_{-\infty}^{\infty} e^{(x-1)}\cos(\pi/2(x-5))\delta(x-3)dx$

More Examples : Consider the signal $x(t)$ below

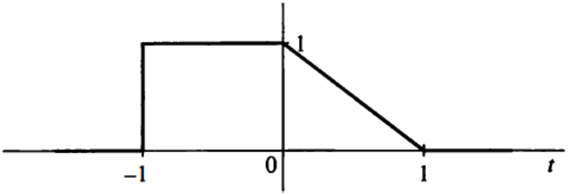
1. sketch $v(t) = 3x(-1/2(t+1))$
2. determine the energy and power of $v(t)$
3. sketch the even part of $v(t)$



More Examples :

Consider the signal $-1/2x(-3t+2)$ shown below

1. Sketch the original signal $x(t)$
2. Sketch the even part of $x(t)$
3. Sketch the odd part of $x(t)$



More Examples :

Suppose that

$$x(t) = \begin{cases} 1 & 0 < t < 1, \\ 0 & otherwise \end{cases}$$

and $h(t) = x(t/a)$, where $0 < a \leq 1$.

1. Determine and sketch $y(t) = x(t) * h(t)$
2. If $dy(t)/dt$ contains only three discontinuities, what is the value of a?

More Examples :

Problem: Let $x(t) = u(t-3) - u(t-5)$ and $h(t) = e^{-3t}u(t)$.

1. Compute $y(t) = x(t) * h(t)$
2. Compute $g(t) = dx(t)/dt * h(t)$
3. How is $g(t)$ related to $y(t)$.

$$\delta(t) \longleftrightarrow h(t) \qquad \delta(t-t_0) \longleftrightarrow h(t-t_0)$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \longleftrightarrow \int_{-\infty}^{t} h(\tau)d\tau \qquad u(t-t_0) \longleftrightarrow \int_{-\infty}^{t-t_0} h(\tau)d\tau$$

$$x(t) = \int_{-\infty}^{t} x(t-\tau)\delta(\tau)d\tau \;\; \frac{d}{dt}x(t) \longleftrightarrow \int_{-\infty}^{t} \frac{d}{dt}x(t-\tau)h(\tau)d\tau$$

$$g(t) = \frac{d}{dt}\int_{-\infty}^{t} x(t-\tau)h(\tau)d\tau$$

$$g(t) = \frac{d}{dt}y(t) \text{ and } y(t) = \frac{1}{3}\left(e^{-3(t-5)} - e^{-3(t-3)}\right)$$

Problem: Let

$$y(t) = e^{-t}u(t) * \sum_{k=-\infty}^{\infty} \delta(t-3k)$$

Show that $y(t) = Ae^{-t}u(t)$ for $0 \leq t < 3$, and determine the value of A.

In the interval $0 \leq t < 3$

$y(t) = \sum_{k=-\infty}^{0} e^{-(t-3k)}u(t) = e^{-t}\sum_{k=-\infty}^{0} e^{3k}u(t) = e^{-t}\sum_{k=0}^{\infty} e^{-3k}u(t)$

$y(t) = \frac{1}{1-e^{-3}}e^{-t}u(t)$

More Examples :

Consider the linear time invariant system S and a signal $x(t) = 2e^{-3t}u(t-1)$. If

$$x(t) \longrightarrow y(t)$$

and

$$dx(t)/dt \longrightarrow -3y(t) + e^{-2t}u(t)$$

determine the impulse response $h(t)$ of S.

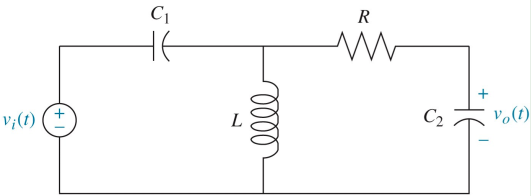
State-Space Approach: $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}u(t)$

n^{th}Order ODE with constant coefficients $a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{d^{n-1}} + \cdots + a_0 y = g(t)$

$$
\begin{aligned}
y &= x_1 \\
\frac{dy}{dt} &= x_2 = \frac{dx_1}{dt} \\
\frac{d^2 y}{dt^2} &= x_3 = \frac{dx_2}{dt} \\
\vdots & \quad \vdots \quad \vdots \\
\frac{d^{n-1} y}{dt^{n-1}} &= x_n = \frac{dx_{n-1}}{dt} \\
\frac{d^n y}{dt^n} &= \frac{dx_n}{dt} = \frac{1}{a_n} g(t) - \frac{a_{n-1}}{a_n} x_n - \cdots - \frac{a_0}{a_n} x_1
\end{aligned}
$$

State-space form :

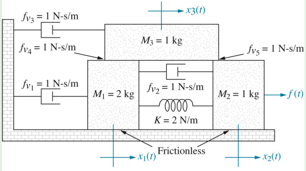
$$
\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & \cdots & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \frac{1}{a_n} g(t)
$$



Since $V_L = L\dot{i}_L$, $i_{C_1} = C_1\dot{v}_{C_1}$ and $i_{C_2} = C_2\dot{v}_o$
The state variables are chosen as v_o, v_{C_1} and i_L.
Eq. 1 : $\dot{v}_o = \frac{1}{C_2} i_R = \frac{1}{C_2 R}(v_i - v_{C_1} - v_o)$
Eq. 2 : $\dot{v}_{C_1} = \frac{1}{C1}(i_L + i_R) = \frac{1}{C1}(i_L + \frac{v_i - v_{C_1} - v_o}{R})$
Eq. 3 : $v_i = v_{C_1} + L\dot{i}_L$

The state-space system is

$$\begin{bmatrix} \dot{v}_o \\ \dot{v}_{C_1} \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC_2} & -\frac{1}{RC_2} & 0 \\ -\frac{1}{RC_1} & -\frac{1}{RC_1} & \frac{1}{C_1} \\ 0 & -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_o \\ v_{C_1} \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_2} \\ \frac{1}{RC_1} \\ \frac{1}{L} \end{bmatrix} v_i$$

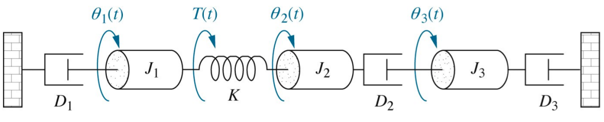


$$\text{Eq. 1}: 0 = M_1\ddot{x}_1 + (f_{v_1} + f_{v_2} + f_{v_4})\dot{x}_1 + Kx_1 - f_{v_2}\dot{x}_2 - Kx_2 - f_{v_4}\dot{x}_3$$
$$\text{Eq. 2}: f(t) = M_2\ddot{x}_2 + (f_{v_2} + f_{v_5})\dot{x}_2 + Kx_2 - f_{v_2}\dot{x}_1 - Kx_1 - f_{v_5}\dot{x}_3$$
$$\text{Eq. 3}: 0 = M_3\ddot{x}_3 + (f_{v_3} + f_{v_4} + f_{v_5})\dot{x}_3 - f_{v_4}\dot{x}_1 - f_{v_5}\dot{x}_2$$

State variables are : $\theta_1 = x_1,\ \theta_2 = \dot{x}_1,\ \theta_3 = x_2,\ \theta_4 = \dot{x}_2,\ \theta_5 = x_3,\ \theta_6 = \dot{x}_3,$

State space system

$$\begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_4 & \dot{\theta}_5 & \dot{\theta}_6 \end{bmatrix}^T$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{K}{M1} & -\frac{(f_{v_1}+f_{v_2}+f_{v_4})}{M_1} & \frac{K}{M_1} & \frac{f_{v_2}}{M_1} & 0 & \frac{f_{v_4}}{M_1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{K}{M_2} & \frac{f_{v_2}}{M_2} & -\frac{K}{M_2} & -\frac{f_{v_2}+f_{v_5}}{M_2} & 0 & \frac{f_{v_5}}{M_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{f_{v_4}}{M_3} & 0 & \frac{f_{v_5}}{M_3}0 & 0 & -\frac{f_{v_3}+f_{v_4}+f_{v_5}}{M_3} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \\ 0 \\ 0 \end{bmatrix} f(t)$$



Eq. 1: $T = J_1\ddot{\theta}_1 + D_1\dot{\theta}_1 + K\theta_1 - K\theta_2$
Eq. 2: $0 = J_2\ddot{\theta}_2 + D_2\dot{\theta}_2 + K\theta_2 - K\theta_1 - D_2\dot{\theta}_3$
Eq. 3: $0 = J_3\ddot{\theta}_3 + (D_2 + D_3)\dot{\theta}_3 - D_2\dot{\theta}_2$
State variables are : $x_1 = \theta_1,\ x_2 = \dot{\theta}_1,\ x_3 = \theta_2,\ x_4 = \dot{\theta}_2,\ ,\ x_5 = \theta_3,\ x_6 = \dot{\theta}_3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{K}{J_1} & -\frac{K}{J_1} & 0 & -\frac{D_2}{J_2} & 0 & \frac{D_2}{J_2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{K}{J_2} & 0 & -\frac{K}{J_2} & -\frac{D_2}{J_2} & 0 & \frac{D_2}{J_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{D_2}{J_3} & 0 & -\frac{D_2+D_3}{J_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} T$$

Matrix Differential Equations: $\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$

Homegenous Solution : $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$
Total solution:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau + e^{\mathbf{A}t}\mathbf{x}(0)$$

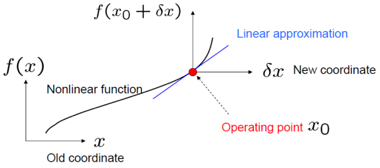
$$\frac{d}{dt}\mathbf{x}(t) = \begin{bmatrix} 0 & 2 \\ -2 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1$ and $\lambda_2 = -4$ so

$$e^{\mathbf{A}t} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} 1/3$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 4/3e^{-t} - 1/3e^{-4t} & 2/3e^{-t} - 2/3e^{-4t} \\ -2/3e^{-t} + 2/3e^{-4t} & -1/3e^{-t} + 4/3e^{-4t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 10/3e^{-t} - e^{-2t} - 4/3e^{-4t} \\ -5/3e^{-t} + e^{-2t} + 8/3e^{-4t} \end{bmatrix}$$



Taylor Series Expansion:

$$f(\delta x + x_0) = f(x_0) + \frac{f'(x_0)}{1!}\delta x + \frac{f''(x_0)}{2!}(\delta x)^2 + \dots$$

Nonlinear system: $\dot{x} = f(x, u)$

u_0 : input $\longrightarrow x_0$: *state*

Perturbation : $u = u_0 + \delta u \longrightarrow x = x_0 + \delta x$

$$f(x, u) = f(x_0, u_0) + \frac{\delta f(x,u)}{\delta x}\delta x\bigg|_{x=x_0, u=u_0} + \frac{\delta f(x,u)}{\delta u}\delta u\bigg|_{x=x_0, u=u_0} +$$

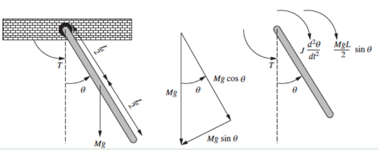
higher order terms

Linearize the system $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \cos(x) = 0$ around $x = \frac{\pi}{4}$

$$\frac{d^2(\delta x + \frac{\pi}{4})}{dt^2} + 2\frac{d(\delta x + \frac{\pi}{4})}{dt} + \cos(\delta x + \frac{\pi}{4}) = 0$$

$$\frac{d^2\delta x}{dt^2} + 2\frac{\delta x}{dt} + \cos(\delta x)\cos(\frac{\pi}{4}) - \sin(\delta x)\sin(\frac{\pi}{4}) = 0$$

$$\frac{d^2\delta x}{dt^2} + 2\frac{\delta x}{dt} - \frac{\sqrt{2}\delta x}{2} = -\frac{\sqrt{2}}{2}$$



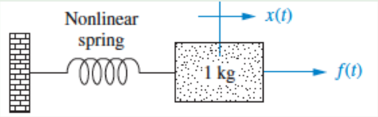
$$J\frac{d^2\theta}{dt^2} + \frac{MgL}{2}\sin(\theta) = T$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{MgL}{2J}\sin(x_1) + \frac{T}{J} \end{aligned}$$

Linearizing around 0 $\longrightarrow x_1 = 0 + \delta x_1$

$$\begin{aligned} \dot{\delta x}_1 &= \delta x_2 \\ \dot{\delta x}_2 &= -\frac{MgL}{2J}\delta x_1 + \frac{T}{J} \end{aligned}$$

Linearize the nonlinear spring system around $f(t) = 10$



$f(t) = 2x^2(t)$ and $f(t) = 10 + \delta f(t)$

Answer : $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4\sqrt{5} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta f(t)$