Boundary Value Problems
Lecture Notes

BM 531
Numerical Methods and C/C++ Programming

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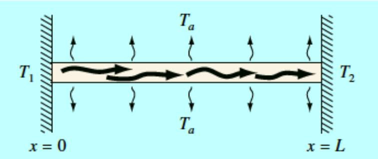
Temperature distribution on a noninsulating rod



$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0$$

$$T(0) = T_1 \text{ and } T(L) = T_2$$

Shooting Method

$L = 10m, \; T_a = 20, \; T_1 = 40, \; T_2 = 200$ and $h' = 0.01$
The analytical solution is $T(x) = 73.45e^{0.1x} - 53.45e^{-0.1x} + 20$

$$\frac{dT}{dx} = z$$

$$\frac{dz}{dx} = -h'(T_a - T)$$

i) Assume a value for $z(0)$–say $z(0) = 10$.
ii) Solve the system of Eq.s using RK4 with *i.e.* $\Delta x = 2$.
iii) $T(10) = 168.38 < 200$.
iv) Make another guess *i.e.* $z(0) = 20$ and repeat ii).
v) $T(10) = 285.90 > 200$.
vi) Perform linear interpolation $z(0) = 10 + \frac{20-10}{285.90-168.38}$
vii) Repeat ii) using $z(0) = 12.69$ to find the correct solutic

Finite Difference Methods

Dirichlet Boundary Condition: $T(0) = 40,\ T(10) = 200$

$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0 \qquad \frac{d^2T}{dx^2} \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} \longrightarrow \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} - h'(T_a - T) = 0$$

$$-T_{i-1} + (2 + h'\Delta x^2)T_i - T_{i+1} = h'\Delta x^2 T_a$$

$$\begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{bmatrix}$$

$$T = \begin{bmatrix} 65.97 & 93.78 & 124.54 & 159.48 \end{bmatrix}^T$$

Finite Difference Methods

Neumann Boundary Condition: $\frac{dT}{dx}(0) = T_a', \; T(L) = T_b$

$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0 \qquad \frac{dT}{dx}(0) = \frac{T_1 - T_{-1}}{2\Delta x} \longrightarrow T_{-1} = T_1 - 2\Delta x \frac{dT}{dx}(0)$$

$$-T_{i-1} + (2 + h'\Delta x^2)T_i - T_{i+1} = h'\Delta x^2 T_a$$

The equation for the left boundary : $\quad -T_{-1} + (2 + h'\Delta x^2)T_0 - T_1 = h'\Delta x^2 T_a$
Using $T_{-1} = T_1 - 2\Delta x \frac{dT}{dx}$ we get

$$-T_1 + 2\Delta x \frac{dT}{dx}(0) + (2 + h'\Delta x^2)T_0 - T_1 = h'\Delta x^2 T_a$$

$$(2 + h'\Delta x^2)T_0 - 2T_1 = h'\Delta x^2 T_a - 2\Delta x \frac{dT}{dx}(0)$$

$$(2 + h'\Delta x^2)T_0 - 2T_1 = h'\Delta x^2 T_a - 2\Delta x T_a'$$

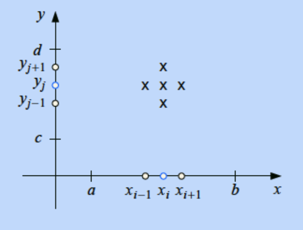
The linear set of equations are solved for $T = [T(0)\; T(2)\; T(4)\; T(6)\; T(8)\; T(10)]$.

Partial Differential Equations

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

- $B^2 - 4AC < 0 \rightarrow$ Elliptic
- $B^2 - 4AC = 0 \rightarrow$ Parabolic
- $B^2 - 4AC > 0 \rightarrow$ Hyperbolic

Finite Difference Method



$$\frac{\partial u}{\partial x} \approx \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x},$$

$$\frac{\partial u}{\partial y} \approx \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i+1j} - 2Tij + T_{i-1j}}{(\Delta x)^2}$$

$$\frac{\partial^2 T}{\partial y^2} \approx \frac{T_{i+1j} - 2Tij + T_{i-1j}}{(\Delta y)^2}$$

Laplace Equation

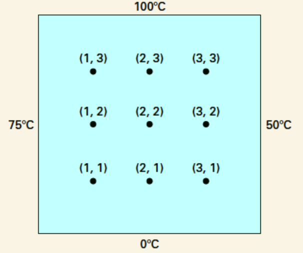
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Poisson's Equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y)$$

where $f(x, y)$ denotes the heat sources and sinks.

Laplacian Equations $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ with Dirichlet Boundary Conditions



$$\frac{T_{i+1j} - 2T_{ij} + T_{i-1j}}{(\Delta x)^2} + \frac{T_{i+1j} - 2T_{ij} + T_{i-1j}}{(\Delta y)^2} = 0$$

If we choose $\Delta x = \Delta y$

$$T_{i+1j} + T_{i-1j} + T_{ij+1} + T_{ij-1} - 4T_{ij} = 0$$

Using boundary conditions for node (1,1) :

$$-T_{11} + T_{12} + T_{21} = -75$$

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{21} \\ T_{31} \\ T_{12} \\ T_{22} \\ T_{32} \\ T_{13} \\ T_{23} \\ T_{33} \end{bmatrix} = \begin{bmatrix} 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150 \end{bmatrix}$$

Gauss-Seidel (Liebmann's) Method

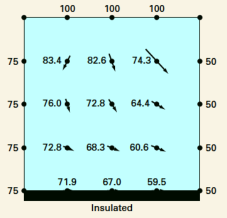
$$T_{ij} = \frac{1}{4}(T_{i+1j} + T_{i-1j} + T_{ij+1} + T_{ij-1})$$

Overrelaxation to speed up Convergence

$$T_{ij}^{new} = \lambda T_{ij}^{new} + (1-\lambda) T_{ij}^{old}$$

λ is a weight factor between 1 and 2.

Laplace Equation Problem with Insulated Edge



$$T_{1j} + T_{-1j} + T_{0j+1} + T_{0j-1} - 4T_{0j} = 0$$

$i = 0, j \longrightarrow$

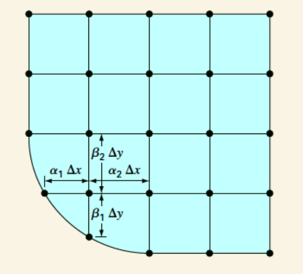
$$\frac{\partial T}{\partial x}(0,j) \approx \frac{T_{1j} - T_{-1j}}{2\Delta x} \rightarrow T_{-1j} = T_{1j} - 2\Delta x \frac{\partial T}{\partial x}(0,j)$$

$$2T_{1j} - 2\Delta x \frac{\partial T}{\partial x} + T_{0j+1} + T_{0j-1} - 4T_{0j} = 0$$

Insulation means $\frac{\partial T}{\partial y} = 0$

$$\begin{bmatrix} 4 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} T_{10} \\ T_{20} \\ T_{30} \\ T_{11} \\ T_{21} \\ T_{31} \\ T_{12} \\ T_{22} \\ T_{32} \\ T_{13} \\ T_{23} \\ T_{33} \end{bmatrix} = \begin{bmatrix} 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150 \end{bmatrix}$$

Irregular Boundaries



$$\left(\frac{\partial T}{\partial x}\right)_{i-1,i} \approx \frac{T_{ij}-T_{i-1j}}{\alpha_1 \Delta x} \text{ and } \left(\frac{\partial T}{\partial x}\right)_{i,i+1} \approx \frac{T_{i+1j}-T_{ij}}{\alpha_2 \Delta x}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) \approx \frac{(\frac{\partial T}{\partial x})_{i\,i+1} - (\frac{\partial T}{\partial x})_{i-1\,i}}{\frac{\alpha_1 \Delta x + \alpha_2 \Delta x}{2}}$$

$$\approx 2\frac{\left[\frac{T_{i-1j}-T_{ij}}{\alpha_1 \Delta x} + \frac{T_{i+1j}-T_{ij}}{\alpha_2 \Delta x}\right]}{(\alpha_1+\alpha_2)\Delta x}$$

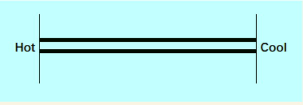
$$\frac{\partial^2 T}{\partial x^2} \approx \frac{2}{(\Delta x)^2}\left[\frac{T_{i-1j}-T_{ij}}{\alpha_1(\alpha_1+\alpha_2)} + \frac{T_{i+1j}-T_{ij}}{\alpha_2(\alpha_1+\alpha_2)}\right]$$

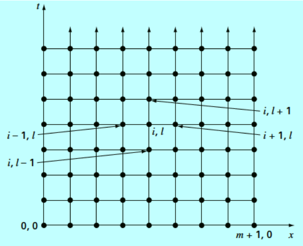
$$\frac{\partial^2 T}{\partial y^2} \approx \frac{2}{(\Delta x)^2}\left[\frac{T_{ij-1}-T_{ij}}{\beta_1(\beta_1+\beta_2)} + \frac{T_{ij+1}-T_{ij}}{\beta_2(\beta_1+\beta_2)}\right]$$

$$\frac{2}{(\Delta x)^2}\left[\frac{T_{i-1j}-T_{ij}}{\alpha_1(\alpha_1+\alpha_2)} + \frac{T_{i+1j}-T_{ij}}{\alpha_2(\alpha_1+\alpha_2)}\right] + \frac{2}{(\Delta y)^2}\left[\frac{T_{ij-1}-T_{ij}}{\beta_1(\beta_1+\beta_2)} + \frac{T_{ij+1}-T_{ij}}{\beta_2(\beta_1+\beta_2)}\right] = 0$$

Finite Difference for Parabolic Equations

$K\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$
Time as well as space is involved.





Explicit Methods

$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2}$, $\frac{\partial T}{\partial t} = \frac{T_i^{l+1} - T_i^l}{\Delta t} \rightarrow \quad T_i^{l+1} = T_i^l + \lambda(T_{i+1}^l - 2T_i^l + T_{i-1}^l)$,
where $\lambda = k\Delta t/(\Delta x)^2$. This is written for all interior nodes.

A horizontal bar with length 10cm. $\Delta x = 2$cm , $\Delta t = 0.1$s.
At $t = 0$ $T(x) = 0$, $T(0) = 100^0 C$ and $T(10) = 50^0 C$
$K = 0.835$, $\lambda = K\Delta t/(\Delta x)^2 = 0.02$
$T_i^{i+1} = T_i^l + \lambda(T_{i+1}^l - 2T_i^l - T_{i-1}^l)$
$T_1^1 = T_1^0 + 0.02(T_2^0 - 2T_1^0 - T_0^0) = 0 + 0.02(0 - 2 \times 0 + 100)$
Convergence $\Rightarrow$ As $\Delta x \to 0$ and $\Delta t \to 0$
finite difference results should approach to the true solution.
Stability $\Rightarrow$ Errors should diminish rather than amplify as the iteration goes on if
$\Delta t \leq \frac{1}{2}\frac{(\Delta x)^2}{K}$.

$\Delta t = 0.1 sec\ K = 0.835\text{cm}^2/\text{s},\ \Delta x = \Delta y = 10\text{cm}.\ \lambda = 10 \times 0.835/(10)^2 = 0.021$

At $t = 0.1sec$ and for $x = 2cm$
$T_1^1 = 0 + 0.021[0 - 2(0) + 100] = 2.0875$
At interior points, $x = 4, 6, 8cm$, the results are

$$
\begin{aligned}
T_2^1 &= 0 + 0.021[0 - 2(0) + 0] &&= 0\\
T_3^1 &= 0 + 0.021[0 - 2(0) + 0] &&= 0\\
T_4^1 &= 0 + 0.021[50 - 2(0) + 0] &&= 1.043
\end{aligned}
$$

At $t = 0.2sec$, the values at the four interior nodes are computed as

$$
\begin{aligned}
T_1^2 &= 2.01 + 0.021[0 - 2(2.01) + 100] &&= 4.09\\
T_2^2 &= 0 + 0.021[0 - 2(0) + 2.09] &&= 0.044\\
T_3^2 &= 0 + 0.021[1.04 - 2(0) + 0] &&= 0.022\\
T_4^2 &= 1.0438 + 0.021[50 - 2(1.04) + 0] &&= 2.04
\end{aligned}
$$

An Implicit Method

$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}$ Spatial derivative is evaluated at time $l+1$

$K\frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2} = \frac{T_i^{l+1} - T_i^l}{\Delta t} \Rightarrow -\lambda T_{i-1}^{l+1} + (1+2\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = T_i^l$

Boundary temperature varies as $T_0^{l+1} = f_0(t^{l+1})$ and $T_{m+1}^{l+1} = f_{m+1}(t^{l+1})$.

$1 + 2 \times 0.2 T_1^1 - 0.02 T_2^1 = T_1^0 + 0.02 \times 100 \rightarrow$

$1.04 T_1^1 - 0.02 T_2^1 = 0 + 0.02 \times 100 = 2.01$

$$\begin{bmatrix} 1.04 & -0.021 & 0 & 0 \\ -0.021 & 1.04 & -0.021 & 0 \\ 0 & -0.021 & 1.04 & -0.021 \\ 0 & 0 & -0.021 & 1.04 \end{bmatrix} \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_1^4 \end{bmatrix} = \begin{bmatrix} 2.01 \\ 0 \\ 0 \\ 1.04 \end{bmatrix}$$

For the next iteration $\begin{bmatrix} 2.01 \\ 0 \\ 0 \\ 1.04 \end{bmatrix} \longrightarrow \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{bmatrix} = \begin{bmatrix} 2.004 \\ 0.041 \\ 0.021 \\ 1.002 \end{bmatrix}$

Crank Nicholson Method

$\frac{\partial T}{\partial t} \approx \frac{T_i^{l+1} - T_i^l}{\Delta t}$, $\frac{\partial T}{\partial t}$ is approximated at $t = l + 1/2$.

$\frac{\partial^2 T}{\partial x^2} = \frac{1}{2}[\frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}]$

$$-\lambda T_{i-1}^{l+1} + 2(1+\lambda) T_i^{l+1} - \lambda T_{i+1}^{l+1} = \lambda T_{i-1}^l + 2(1-\lambda) T_i^l + \lambda T_{i+1}^l$$

Boundary temperature varies as
$T_0^{l+1} = f_0(t^{l+1})$ and $T_{m+1}^{l+1} = f_{m+1}(t^{l+1})$.

1st interior node:
$2(1+\lambda) T_1^{l+1} - \lambda T_2^{l+1} = \lambda f_0(t^l) + 2(1-\lambda) T_1^l + \lambda T_2^l + \lambda f_0(t^{l+1})$

Last interior node:
$-\lambda T_{m-1}^{l+1} + 2(1+\lambda) T_m^{l+1} = \lambda f_{m+1}(t^l) + 2(1-\lambda) T_m^l + \lambda T_{m-1}^l + \lambda f_m(t^{l+1})$

Parabolic Equations in two spatial dimensions

$K(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) = \frac{\partial T}{\partial t}$,

Stability Criterion $\Delta t \leq \frac{1}{8}\frac{(\Delta x)^2+(\Delta y)^2}{K}$

Alternating Direction Implicit Scheme

Explicit

$$K[\frac{T^l_{i+1j}-2T^l_{ij}+T^l_{i-1j}}{(\Delta x)^2} + \frac{T^{l+1/2}_{ij+1}-2T^{l+1/2}_{ij}+T^{l+1/2}_{ij-1}}{(\Delta y)^2}] = \frac{T^{l+1/2}_{ij}-T^l_{ij}}{\Delta t/2}$$

$$\lambda T^{l+1/2}_{ij-1} + (1+\lambda)T^{l+1/2}_{ij} - \lambda T^{l+1/2}_{ij+1} = \lambda T^l_{i-1j} + 2(1-\lambda)T^l_{ij} + \lambda T^l_{i+1j}$$

Implicit

$$K[\frac{T^{l+1}_{i+1j}-2T^{l+1}_{ij}+T^{l+1}_{i-1j}}{(\Delta x)^2} + \frac{T^{l+1/2}_{ij+1}-2T^{l+1/2}_{ij}+T^{l+1/2}_{ij-1}}{(\Delta y)^2}] = \frac{T^{l+1}_{ij}-T^{l+1/2}_{ij}}{\Delta t/2}$$

$$\lambda T^{l+1/2}_{i-1j} + 2(1+\lambda)T^{l+1}_{ij} - \lambda T^{l+1}_{i+1j} = \lambda T^{l+1/2}_{ij-1} + 2(1-\lambda)T^{l+1/2}_{ij} + \lambda T^{l+1/2}_{ij+1}$$

A heat conduction problem on a $40 \times 40cm$ plate

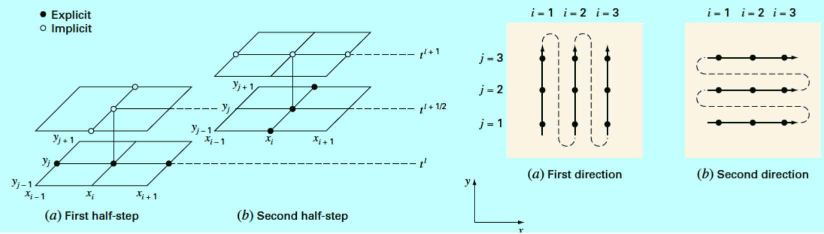
$\Delta x = 10cm$, $\lambda = 0.835(10)/(10)^2 = 0.0835$,
At $t = 5sec$

$$\begin{bmatrix} 2.17 & -0.084 & 0 \\ -0.084 & 2.167 & -0.0840 \\ 0 & -0.084 & 2.17 \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \end{bmatrix} = \begin{bmatrix} 6.26 \\ 6.26 \\ 14.61 \end{bmatrix}$$

$\begin{bmatrix} T_{21} \\ T_{22} \\ T_{23} \end{bmatrix}$ and $\begin{bmatrix} T_{31} \\ T_{32} \\ T_{33} \end{bmatrix}$ are also computed similarly.
At $t = 10sec$

$$\begin{bmatrix} 2.17 & -0.084 & 0 \\ -0.084 & 2.167 & -0.084 \\ 0 & -0.084 & 2.17 \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{21} \\ T_{31} \end{bmatrix} = \begin{bmatrix} 12.06 \\ 0.26 \\ 8.06 \end{bmatrix}$$

$\begin{bmatrix} T_{12} \\ T_{22} \\ T_{32} \end{bmatrix}$ and $\begin{bmatrix} T_{13} \\ T_{23} \\ T_{33} \end{bmatrix}$ are also computed similarly.



At $t = 5$s,

Apply $\longrightarrow \lambda T_{ij-1}^{l+1/2} + (1+\lambda) T_{ij}^{l+1/2} - \lambda T_{ij+1}^{l+1/2} = \lambda T_{i-1j}^{l} + 2(1-\lambda) T_{ij}^{l} + \lambda T_{i+1j}^{l}$

$$\begin{bmatrix} 2.1670 & -0.0835 & 0.0000 \\ -0.0835 & 2.1670 & -0.0835 \\ 0.0000 & -0.0835 & 2.1670 \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \end{bmatrix} = \begin{bmatrix} 6.26 \\ 6.26 \\ 14.61 \end{bmatrix}$$ repeat for each column.

At $t = 10$s,

Apply $\longrightarrow \lambda T_{i-1j}^{l+1/2} + 2(1+\lambda) T_{ij}^{l+1} - \lambda T_{i+1j}^{l+1} = \lambda T_{ij-1}^{l+1/2} + 2(1-\lambda) T_{ij}^{l+1/2} + \lambda T_{ij+1}^{l+1/2}$

$$\begin{bmatrix} 2.1670 & -0.0835 & 0.0000 \\ -0.0835 & 2.1670 & -0.0835 \\ 0.0000 & -0.0835 & 2.1670 \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{21} \\ T_{31} \end{bmatrix} = \begin{bmatrix} 12.64 \\ 0.26 \\ 8.06 \end{bmatrix}$$ repeat for each row.