Root Finding
Lecture Notes

BM 531
Numerical Methods and C/C++ Programming

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Solution of Nonlinear Equations $f(x) = 0$

Fixed Point Iteration

p_0 : Initial Point

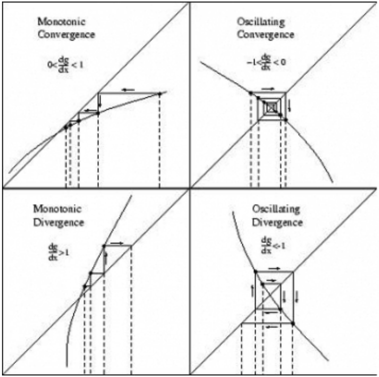
$p_1 = f(p_0)$

$\vdots$

$p_k = f(p_{k-1})$

$p_{k+1} = f(p_k)$

until $x = f(x)$



Bracketing Methods

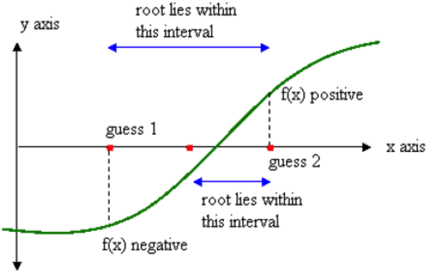
Bisection Method

Find an interval defined by $[a, b]$ determine $c = (a + b)/2$ and analyze the three possibilities

If $f(a)$ and $f(c)$ have opposite signs, a root lies in $[a, c]$.

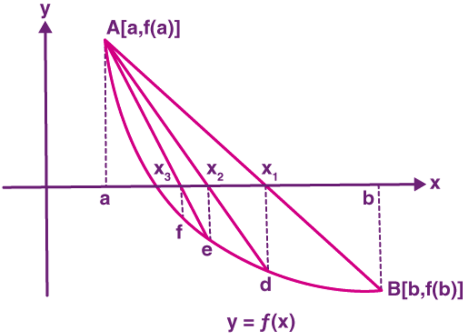
If $f(c)$ and $f(b)$ have opposite signs, a root lies in $[c, b]$.

If $f(c) = 0$ we found a root at $x = c$.



Method of False Position (Regula Falsi)

$m = (f(b) - f(a))/(b - a)$
$m = (0 - f(a))/(x_1 - a)$
$x_1 = b - f(b)(b - a)/(f(b) - f(a))$
$x_n = b_n - f(b_n)(b_n - a_n)/(f(b_n) - f(a_n))$



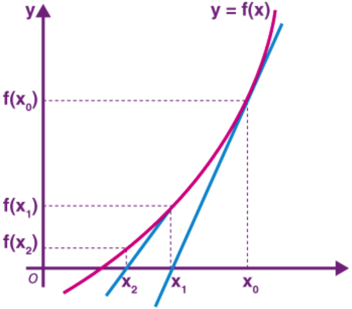
Bracketing Methods

Newton–Raphson Method

$m = (0 - f(x_0))/(x_1 - x_0) = f'(x_0)$

$x_1 = x_0 - f(x_0)/f'(x_0)$

$x_k = x_{k-1} - f(x_{k-1})/f'(x_{k-1})$ for $k = 1, 2, \ldots$



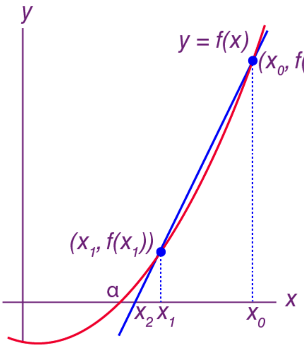
Bracketing Methods

Secant Method

$m = (f(x_1) - f(x_0))/(x_1 - x_0) = (0 - f(x_1))/(x_2 - x_1)$

$x_2 = x_1 - f(x_1)(x_1 - x_0)/(f(x_1) - f(x_0))$

$x_{k+1} = x_k - f(x_k)(x_k - x_{k-1})/(f(x_k) - f(x_{k-1}))$



Iteration for nonlinear systems

$$f1(x,y) = x^2 - 2x - y + 0.5 = 0,$$
$$f2(x,y) = x^2 + 4y^2 - 4 = 0,$$
$$x = (x^2 - y + 0.5)/2 = g_1(x,y),$$
$$y = (-x^2 - 4y^2 + 8y + 4)/8 = g_2(x,y),$$
$$x_{k+1} = (x_k^2 - y_k + 0.5)/2,$$
$$y_{k+1} = (-x_k^2 - 4y_k^2 + 8y_k + 4)/8,$$

Theorem

Let $D = \{(x_1, x_2, \ldots, x_n)t | a_i \leq x_i \leq b_i$, for each $i = 1, 2, \ldots, n\}$ for some collection of constants $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. Suppose G is a continuous function from $D \subset R^n$ into R^n with the property that $G(\mathbf{x}) \in D$ whenever $\mathbf{x} \in D$. Then G has a fixed point in D. Moreover, suppose that all the component functions of G have continuous partial derivatives and a constant $K < 1$ exists with $|\frac{\partial g_i(\mathbf{x})}{\partial x_j}| \leq K/n$, whenever $\mathbf{x} \in D$, for each $j = 1, 2, \ldots, n$ and each component function g_i. Then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by an arbitrarily selected $\mathbf{x}(0)$ in D and generated by $\mathbf{x}^{(k)} = G(\mathbf{x}^{(k-1)})$, for each $k \geq 1$ converges to the unique fixed point $p \in D$ and $\left|| \mathbf{x}^{(k)} - \mathbf{p} \right||_\infty \leq \frac{K^k}{1-K} || \mathbf{x}^{(1)} - \mathbf{x}(0) ||_\infty$.

Example

Assume there is a unique solution for $\mathbf{x} = G(\mathbf{x})$ in the interval $D = \{(x_1, x_2, x_3) | -1 \leq x_i \leq 1,\ i = 1, 2, 3\}$ for the system below.

$$
\begin{aligned}
3x_1 - \cos(x_2 x_3) - 1/2 &= 0 \\
x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 &= 0 \\
e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} &= 0
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= \frac{1}{3}\cos(x_2 x_3) + \frac{1}{6} \\
x_2 &= \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1 \\
x_3 &= -\frac{1}{20} e^{-x_1 x_2} + -\frac{10\pi - 3}{60}
\end{aligned}
$$

$$
\begin{array}{llll}
|g_1| \leq 0.5 & |\frac{\partial g_1}{\partial x_1}| = 0 & |\frac{\partial g_1}{\partial x_2}| < 0.281 & |\frac{\partial g_1}{\partial x_3}| < 0 \\
|g_2| \leq 0.09 & |\frac{\partial g_2}{\partial x_1}| < 0.238 & |\frac{\partial g_2}{\partial x_2}| = 0 & |\frac{\partial g_2}{\partial x_3}| = 0.119 \\
|g_3| \leq 0.61 & |\frac{\partial g_3}{\partial x_1}| < 0.119 & |\frac{\partial g_3}{\partial x_2}| < 0.14 & |\frac{\partial g_3}{\partial x_3}| = 0
\end{array}
$$

for $x_1, x_2, x_3 \in D = [-1, 1]$ which indicates that G has a unique fixed point in D and nonlinear system has a solution in D.

Newton's Method

Newton's Method based on linear approximation

$$\mathbf{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ \\ f_2(x,y) \end{bmatrix} = \mathbf{f}(x_0,y_0) + (x-x_0)\partial\mathbf{f}(x,y)/\partial x + (y-y_0)\partial\mathbf{f}(x,y)/\partial y + \ldots$$

Jacobian Matrix

$$\mathbf{J}(x,y) = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix}$$

$\mathbf{f}(x,y) = \mathbf{f}(x_0,y_0) + \mathbf{J}(x,y)[(x-x_0) \quad (y-y_0)]' + \ldots$

Based on iteration

$\mathbf{P}_{k+1} = \mathbf{P}_k + \Delta\mathbf{P} = \mathbf{P}_k - \mathbf{J}(p_k,q_k)^{-1}\mathbf{f}(p_k,q_k)$

where $\mathbf{P}_k = [p_k \quad q_k]'$

Outline of Newton's method:

1. Evaluate the function $\mathbf{f}(\mathbf{P}_k) = \begin{bmatrix} f_1(p_k, q_k) \\ \\ f_2(p_k, q_k) \end{bmatrix}$,
2. Evaluate the Jacobian $\mathbf{J}(\mathbf{P}_k)$,
3. Solve the linear system $\mathbf{J}(\mathbf{P}_k)\Delta\mathbf{P} = -\mathbf{f}(\mathbf{P}_k) \;\; for \quad \Delta\mathbf{P}$,
4. Compute the next point $\mathbf{P}_{k+1} = \mathbf{P}_k + \Delta\mathbf{P}$,
5. Repeat 1 until convergence.