BASIC CONCEPTS

Lecture Notes

Ahmet Ademoglu, *PhD*
Bogazici University
Institute of Biomedical Engineering

BIOMEDICAL INSTRUMENTATION

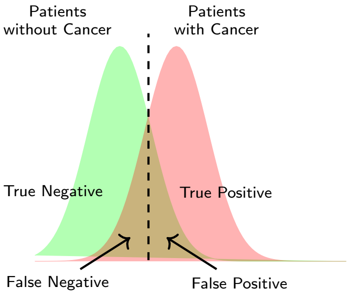
Involves the following disciplines;

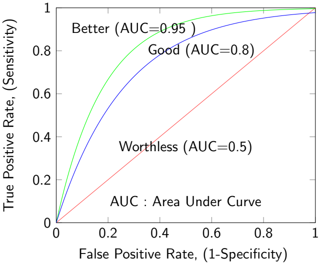
- Systems Theory
- Circuit Theory
- Control Theory
- Signal Processing
- Statistical Analysis
- Digital and Analog Systems Design
- Systems Programming

BASIC STATISTICAL CONCEPTS

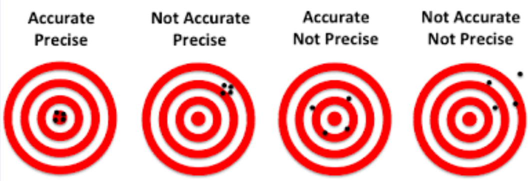
1. Sample Mean: $\overline{x} = \frac{\sum x_i}{n}$
2. Sample Variance and Standard Deviation: $s = \sqrt{\frac{\sum (x_i - \overline{x})^2}{n-1}}$
3. Standard Error of the Mean: $s_{\overline{x}} = \frac{s}{\sqrt{n-1}}$
4. Correlation Coefficient: $r = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum (x_i - \overline{x})}\sqrt{\sum (y_i - \overline{y})}}$
5. $p-$value
6. Specificity : Accept $\mathcal{H}_0$ when it is True (TN) *True Negative*
7. Sensitivity (Power): Reject $\mathcal{H}_0$ when it is False (TP) *True Positive*
8. Type I Error: (FP) *False Positive*
9. Type II Error : (FN) *False Negative*

RECEIVER OPERATING CHARACTERISTICS CURVE





- Accuracy: $100 \times \frac{\textit{True Value} - \textit{Measured Value}}{\textit{True Value}}$
- Precision : 2.346 V is more precise than 2.34 V



- Resolution : The smallest quantity that can be measured
- Reproducibility (Repeatability) : ability to give the same output to the same input after a certain period of time

Self-Study Question

We have two analog to digital converters (ADC), an 8-bit and a 16-bit. Calculate the maximum number of distinct binary values each ADC can produce, and the voltage step that the least significant bit (LSB) represents. Assume that the voltage range is 0 V to 5 V minus one LSB in all cases.

8-bit ADC:

$28 = 256$ distinct values (ranges from 0 to 255).

$\text{LSB} = 5/256 = 19.5$ mV

16-bit ADC:

$216 = 65536$ distinct values (ranges from 0 to 65535).

$\text{LSB} = 5/65536 = 0{:}0763$ mV

Gaussian (Normal) Distribution $\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

μ : mean
σ^2: Variance

Multivariate Normal Distribution :

$\mathcal{N}(\mathbf{x}|\mu,\Sigma) = \left(\frac{1}{2\pi}\right)^{\frac{N}{2}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)}$

N: dimension of $\mathbf{x}$.

Expected value or average of a random variable with distribution $p_X(x)$ is

$$E\{x\} = \int_{-\infty}^{\infty} x p_X(x) dx$$

and its variance is

$$E\{(x - E\{x\})^2\} = E\{x^2\} - E\{x\}^2 = \int_{-\infty}^{\infty} x^2 p_X(x) dx - E\{x\}^2.$$

Self Study Question

We estimate the average value of an independent and identically distributed random variable x whose distribution is $\mathcal{N}(\mu, \sigma^2)$ by using the sample mean estimator *i.e*

$$\bar{x} = \frac{1}{N} \sum_{n=1}^{N} x_n$$

Is the sample mean is unbiased estimator? What is its variance?

$$\mathbb{E}\{\bar{x}\} = \mathbb{E}\left\{\frac{1}{N}\sum_{n=1}^{N} x_n\right\} = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\{x_n\} = \frac{1}{N}\sum_{n=1}^{N}\mu = \mu \text{ unbiased.}$$

Variance of x is

$$\mathbb{E}\{\bar{x}^2\} - \mathbb{E}\{\bar{x}\}^2 = \mathbb{E}\left\{\frac{1}{N}\sum_{n=1}^{N} x_n\right\}^2 - \mu^2 = \frac{1}{N^2}\mathbb{E}\left\{\sum_{n=1}^{N} x_n\right\}^2 - \mu^2$$

$$= \frac{1}{N^2}\mathbb{E}\{(x_1(x_1 + x_2 + \ldots) + x_2(x_1 + x_2 + \ldots) + \ldots)\} - \mu^2$$

$$= \frac{1}{N^2}\{(\sigma^2 + \mu^2 + (N-1)\mu^2) + (\sigma^2 + \mu^2 + (N-1)\mu^2) + \ldots)\} - \mu^2$$

$$= \frac{1}{N^2}\{N(\sigma^2 + \mu^2) + N(N-1)\mu^2\} - \mu^2$$

$$= \frac{1}{N^2}\{N\sigma^2 + \mu^2 N^2\} - \mu^2$$

$$= \frac{1}{N}\sigma^2$$

Bias-Variance Trade-off

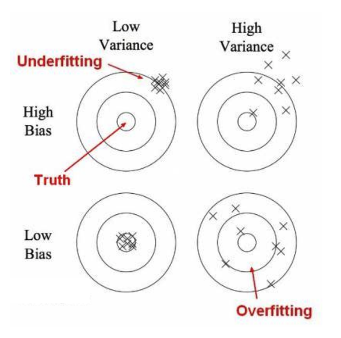
Let's assume we make a measurement such that the independent parameter x is related to our measurements as

$$y = f(x) + \epsilon$$

where the error term often has a normal distribution as $\mathbf{N}(\epsilon|0, \sigma^2)$.

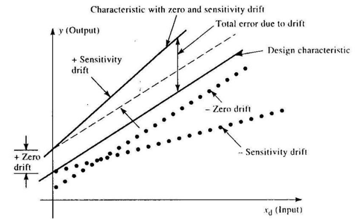
$$Error = E\{(y - \hat{f}(x))^2\} = \underbrace{[f(x) - E\{\hat{f}(x)\}]^2}_{Variance} + \underbrace{f(x) - E\{\hat{f}(x)\}}_{Bias} + \underbrace{\sigma_e^2}_{Irreducible\ Error}$$

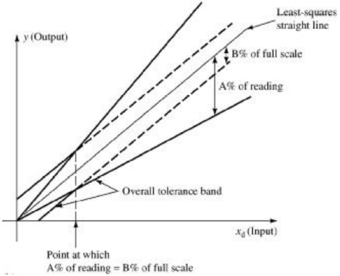
$\hat{f}(x)$: Estimate of y based on a model estimator and the observation x.



- Sensitivity: Change in Output/Change in Input $= \Delta y / \Delta x_d$.
- Zero Drift: fast changes due to motion, hysteresis, vibration, schock etc.
- Sensitivity Drift: due to power supply, nonlinearities, change in ambient conditions etc.
- Linearity, Input Range, Input Impedance.

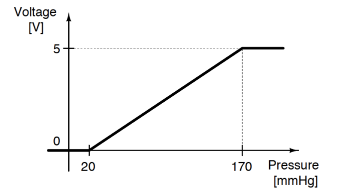
Zero and sensitivity drifts





Self Study Question

You are given a biomedical instrument that measures blood pressure from a cuff at its input and produces a digital reading on an output display. The instrument transduces the pressure into a voltage, and digitizes this voltage by an 8-bit analog-to-digital converter (ADC). The transducer voltage as a function of pressure is shown in the graph below, and the ADC full-scale voltage range is from 1 V to 4 V.



a) Find the sensitivity of the transducer, and the range of pressure over which it operates.

$Sensitivity = \frac{Voltage}{Pressure} = \frac{5-0}{170-20} = \frac{1}{30} V/mmHg$,

$Range = [20 mmHg \text{ to } 170 mmHg]$.

b) Find the resolution of the instrument, and the range of pressure over which it produces a valid reading.

$1V$ output corresponds to $20 + 30 = 50 mmHg$

$4V$ output corresponds to $170 - 30 = 140 mmHg$,

therefore, the range is $[50 mmH\ 140 mmHg]$.

$Resolution = Range/2^8 = 90 mmHg/256 = 0.35 mmHg$.

c) You discover that the transducer for known pressure values produces a voltage that on average is $0.2V$ lower than expected, and with a standard deviation of $0.5V$. Find the accuracy and precision of the instrument, in units mmHg.

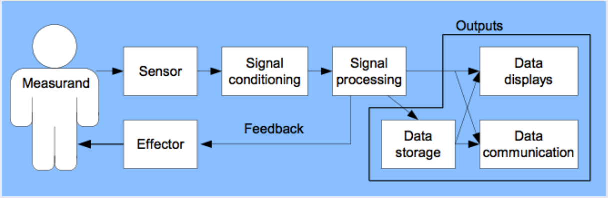
$Accuracy = True\ Value - Mean\ Measured\ Pressure = 0.2V \cdot 30 mmHg/V = 6 mmHg$.

$Precision = Stdev.\ of\ of\ Measured\ Pressure = 0.5V \cdot 30 mmHg/V = 15 mmHg$.

Self Study Question

If you are given two identical copies of such an instrument, how would you use these two to construct a better instrument that produces a consistent zero reading for zero pressure independent of temperature and other environmental variations?
Hint : Use subtraction method.

General Bioinstrumentation System



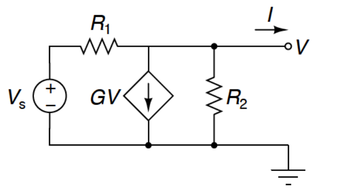
Measurand : Physical quantity, property, or condition that the system measures *biopotential, pressure, flow, dimension (imaging), displacement (velocity, acceleration, and force), impedance, temperature, chemical concentrations*.

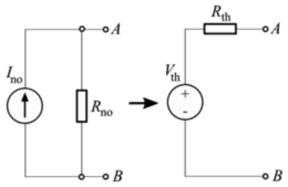
General Bioinstrumentation System

Measurement	Range	Frequency, Hz	Method
Blood flow	1 to 300 mL/s	0 to 20	Electromagnetic or ultrasonic
Blood pressure	0 to 400 mmHg	0 to 50	Cuff or strain gage
Cardiac output	4 to 25 L/min	0 to 20	Fick, dye dilution
Electrocardiography	0.5 to 4 mV	0.05 to 150	Skin electrodes
Electroencephalography	5 to 300 μ V	0.5 to 150	Scalp electrodes
Electromyography	0.1 to 5 mV	0 to 10000	Needle electrodes
Electroretinography	0 to 900 μ V	0 to 50	Contact lens electrodes
pH	3 to 13 pH units	0 to 1	pH electrode
pCO_2	40 to 100 mmHg	0 to 2	pCO_2 electrode
pO_2	30 to 100 mmHg	0 to 2	pO_2 electrode
Pneumotachography	0 to 600 L/min	0 to 40	Pneumotachometer
Respiratory rate	2 to 50 breaths/ min	0.1 to 10	Impedance
Temperature	32 to 40 °C	0 to 0.1	Thermistor

Thevenin and Norton Equivalent Circuits

Derive the Thevenin equivalent at node V in the circuit below:





Thevenin Voltage:

Open circuit voltage $V = V_s - I_1 R_1$,

Kirchoff current law yields $I_1 = GV + I_2$ and $I_2 = V/R_2$.

$V = V_s - (GV + V/R_2)R_1$ or $V_T = V = \frac{R_2}{R_1+R_2+R_1R_2G} V_s$

Thevenin Resistance:

We suppress the voltage source V_s, apply a voltage V_c to the output and measure the current I_c going into the output terminal to find the Thevenin impedance seen from the output port as $R_T = V_c/I_c$.

R_1 becomes parallel to R_2 as a resistance $R = R_1R_2/(R_1 + R_2)$,

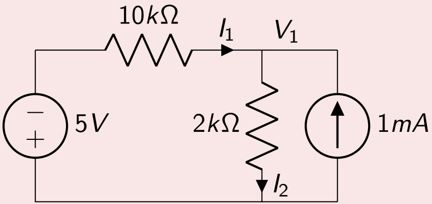
$I_R = V_c/R$, $GV_c + I_R = I_c$ which yields $\frac{V_c}{I_c} = R_T = \frac{R_1R_2}{R_1+R_2+R_1R_2G}$

Power Dissipation in a Circuit

Instantaneous Power $P = V \cdot I$

Average Power with a sinusoidal input with frequency ω

$$P_{Av} = \frac{1}{T}\int_0^T V(t)I(t)dt \text{ where } \frac{2\pi}{T} = \omega$$



What is the total power consumed by the circuit?

$I_1 + 1 \cdot 10^{-3} = I_2$, $I_2 = \frac{V_1}{2 \cdot 10^3}$ and $I_1 = \frac{5 - V_1}{10 \cdot 10^3}$

$10^4 I_1 = 5 - 2 \cdot 10^3 I_2$ or $I_1 = 5 \cdot 10^{-4} - 0.2 I_2$

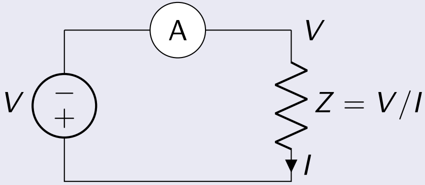
$I_2 = 1.25mA$ and $I_1 = 0.25mA$

$Power_{Total} = Power_{10k\Omega} + Power_{2k\Omega} = 2.5 \cdot 0.25 + 2.5 \cdot 1.25 = 3.75mW$

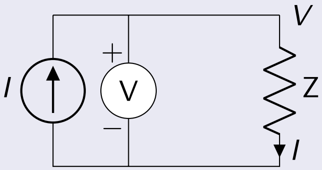
Impedance Measurements

Impedance $Z = \frac{\text{Effort Variable}}{\text{Input Variable}} = \frac{\text{voltage}}{\text{current}}$ or $\frac{\text{pressure}}{\text{flow}}$ or $\frac{\text{temperature}}{\text{heat}} \dots$

Input Impedance Z can be measured by

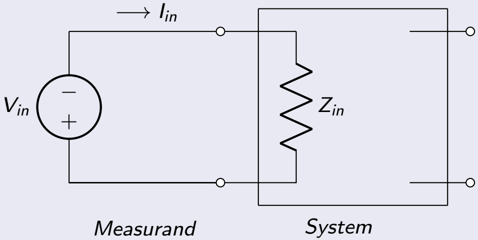


Voltage Source + Ammeter

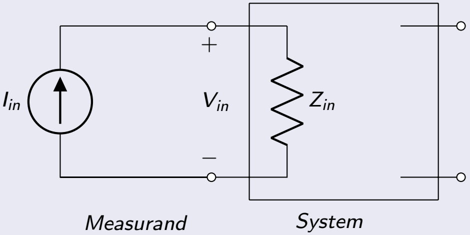


Current Source + Voltmeter

Impedance of Measurement System



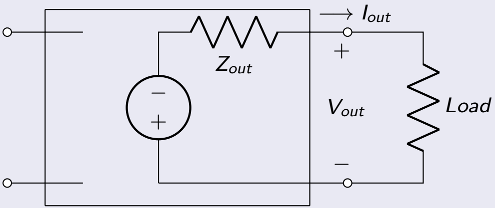
If the measurand is voltage, then Z_{in} must be large so that very little current is drawn from the measurand and voltage measured by the system is almost equal to V_{in}.



If the measurand is current, then Z_{in} must be small so that very little voltage is dropped across the measurand and current measured by the system is almost equal to I_{in}.

Ideally,

- $Z_{in} \approx \infty$ for voltage measurands *i.e.* $I_{in} = 0$ all V_{in} values as for an ideal voltmeter.
- $Z_{in} \approx 0$ for current measurands *i.e.* $V_{in} = 0$ all I_{in} values as for an ideal ammeter.



Output Impedance of System
$Z_{out} = \frac{V_{out}}{V_{in}}$

Ideally,

- $Z_{out} \approx 0$ which yields $\Delta V_{out} = 0$ for all ΔI_{out} values for voltage drivers as an ideal voltage source (Thevenin Equivalent Circuit).
- $Z_{out} \approx \infty$ which yields $\Delta I_{out} = 0$ for all ΔV_{out} values for current drivers as an ideal current source (Norton Equivalent Circuit).

Continuous Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

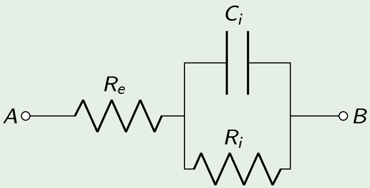
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

Determine the transfer function of the linear system whose input-output is decribed by the linear differential equation $\frac{dy(t)}{dt} + ay(t) = x(t)$ with $a > 0$.

$$H(\omega) = \frac{1}{a + j\omega}$$

$$h(t) = e^{-at}u(t)$$

Self Study Problem



R_e : resistance of the extracellular fluid,
R_i : resistance of lipids (fat) separating extracellular and intracellular fluids
C_i : corresponding capacitance approximating the contribution by cell membranes

Bioelectrical impedance analysis (BIA) is used as a non–invasive method to estimate the percent level of fat in the human body. The impedance of the human body is roughly approximated as the following circuit, where .

1. Write down an expression for the impedance between terminals A and B as a function of radial frequency ω. $Z = \frac{R_i + R_e(1+A^2)}{1+A^2} - j\frac{R_i A}{1+A^2}$ and $A = \omega R_i C_i$.
2. What is the magnitude of the total impedance as a function of ω? $(R_e^2 + (R_i^2 + 2R_e)/(1 + \omega^2 R_i^2 C_i^2))^{1/2}$.
3. What is the phase of the total impedance as a function of ω? $\angle Z = \arctan(-\omega R_i^2 C_i/(R_i + R_e(1 + \omega^2 R_i^2 C_i^2))$
4. ADCs measure only voltage and cannot measure impedance directly. Briefly describe one way to convert an impedance into a voltage.
   Hint: Use voltage divider and show the circuitry.

Continuous Fourier Series of Periodic Signals with period T.

$$x(t) = \sum_{k=\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency.

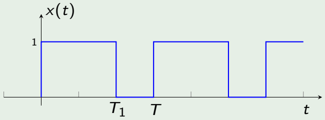
The a_k are determined by

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Specifically,

$$a_0 = \frac{1}{T} \int_T x(t) dt$$

Determine the Fourier Series of $x(t)$.



$$a_k = \frac{1}{T} \int_0^{T_1} 1 \cdot e^{-jk\omega_0 t} dt = -\frac{[e^{-jk\omega_0 T_1} - 1]}{Tjk\omega_0}$$

$$= e^{-jk\omega_0 T_1/2} \frac{[e^{jk\omega_0 T_1/2} - e^{-jk\omega_0 T_1/2}]}{Tjk\omega_0}$$

$$= e^{-jk\pi T_1/T} \frac{\sin(k\pi T_1/T)}{k\pi} = e^{\frac{-jk\pi T_1}{T}} \frac{T_1}{T} sinc(k\frac{T_1}{T})$$

Generalized Dynamic Characteristics

$$a_n \frac{d^n y}{dt^n} + \cdots + a_1 \frac{dy}{dt} + a_0 y(t) = b_m \frac{d^m x}{dt^m} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

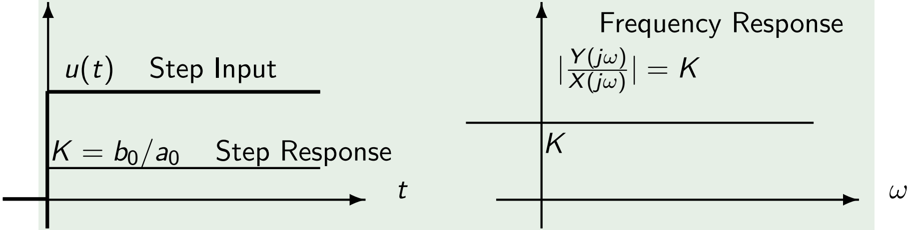
$$\frac{Y(j\omega)}{X(j\omega)} = \frac{b_m (j\omega)^m + \cdots + b_1 (j\omega) + b_0}{a_n (j\omega)^n + \cdots + a_1 (j\omega) + a_0}$$

Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad x(t) = \frac{1}{2\pi j} \oint_{\sigma-\infty}^{\sigma+\infty} X(s) e^{st} ds$$

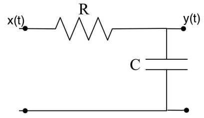
$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

Potentiometer: Zero-order Instrument $\longrightarrow \frac{Y(j\omega)}{X(j\omega)} = \frac{b_0}{a_0} = K$ Static Sensitivity

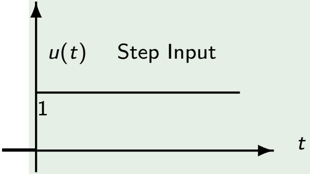


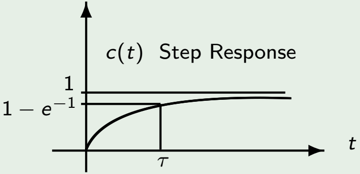
First Order Instrument

$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t) \qquad \frac{Y(j\omega)}{X(j\omega)} = \frac{K}{1+j\omega\tau}$ where $\frac{a_1}{a_0} = \tau = RC$ and $K = \frac{b_0}{a_0}$



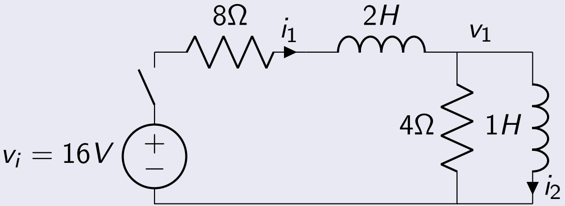
$$\frac{Y(j\omega)}{X(j\omega)} = \frac{K}{\sqrt{(1+\omega^2\tau^2)}} \angle\phi = \tan^{-1}(-\omega\tau)$$





$$\mathcal{L}\{u(t)\} = U(s) = \frac{1}{s} \text{ and } \frac{Y(s)}{X(s} = \frac{1/RC}{1/RC + s} \rightarrow C(s) = H(s)U(s) = \frac{1}{s}\frac{a}{a+s} = \frac{1}{s} - \frac{1}{s+a}$$

$$c(t) = u(t)(1 - e^{-t/RC})$$

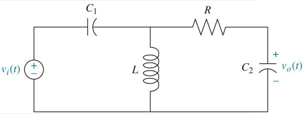


$$V_1(s) = sI_2(s)$$
$$I_1(s) = V_1(s)/4 + I_2(s)$$
$$V_i(s) = (8 + 2s)I_1(s) + V_1(s)$$

$$I_2(s) = V_i(s)\frac{2}{s^2 + 10s + 16} = \frac{32}{s(s^2 + 10s + 16)}$$
$$i_2(t) = 32\left(\frac{1}{16} - \frac{1}{12}e^{-2t} + \frac{1}{48}e^{-8t}\right)u(t)$$

```
t=[-1:0.01:8]'; % time axis
u = 16*0.5*(1+sign(t)); % step function input
B = 2; % nominator polynomial coefficients [1]
A = [1 10 16] ; % denominator polynomial coefficients
y = lsim(B,A,u,t); % matlab function to determine system output
plot(t,[y]);grid;title('Step response' );
```

Find the Transfer Function $\frac{V_o(s)}{V_i(s)}$



Since $i_C(t) = C\frac{dv_C(t)}{dt}$, $Z_C(s) = \frac{V}{I}(s) = \frac{1}{sC}$

$v_L(t) = L\frac{di_L(t)}{dt}$, $Z_L(s) = \frac{V}{I}(s) = sL$

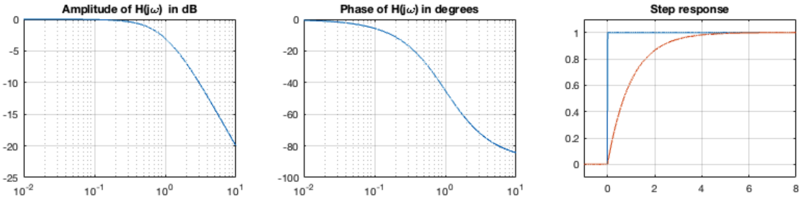
$v_R(t) = Ri(t)$, $Z_R(s) = \frac{V}{I}(s) = R$

Thevenin Impedance $Z_T = \frac{\frac{1}{sC_1}sL}{\frac{1}{sC_1}+sL} + R = \frac{sL+R(1+s^2LC_1)}{1+s^2LC_1}$

Thevenin Voltage $V_T = V_i\frac{sL}{\frac{1}{sC_1}+sL} = V_i\frac{s^2LC_1}{1+s^2LC_1}$,

$$V_o = V_T\frac{\frac{1}{sC_2}}{\frac{1}{sC_2}+Z_T} == V_i\frac{sL}{\frac{1}{sC_1}+sL} = V_i\frac{s^2LC_1}{(1+s^2LC_1)(1+sC_2(\frac{sL+R(1+s^2LC_1)}{1+s^2LC_1}))}$$

$$\frac{V_o}{V_i} = \frac{s^2LC_1}{1+s^2LC_1+sC_2(sL+R(1+s^2LC_1))} = \frac{s^2LC_1}{s^3LC_1+s^2L(C_1+C_2)+sRC_2}$$

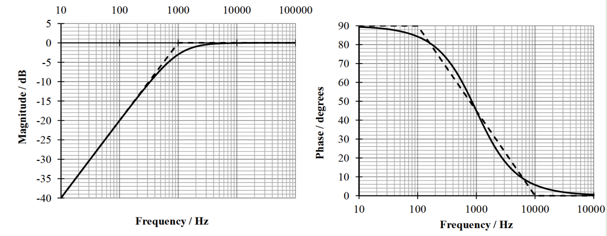


First orderLow pass filter response

```
tau = 1; % time constant
w = [0:0.01:10]; % freqency range in radians
H = ones(size(w))./(i*w*tau +1);
subplot(2,1,1);semilogx(w,20*log10(abs(H)));grid;
title('Amplitude of H(j\omega)  in dB' );
subplot(2,1,2);semilogx(w,phase(H)/pi*180);grid;
title('Phase of H(j\omega) in degrees');
% step response
t=[-1:0.01:8]'; % time axis
u = 0.5*(1+sign(t)); % step function input
B = 1 ; % nominator polynomial coefficients [1]
A = [tau 1] ; % denominator polynomial coefficients [s tau +1]
y = lsim(B,A,u,t); % matlan function to determine system output
subplot(2,2,3);plot(t,[u y]);grid;title('Step response' );axis([-1 8 -.1 1.1])
```

Self Study Question

The Bode plot of a particular first order filter is shown below. The dashed lines are straight line approximations of the actual plot (the solid curves).



1. What kind of filter is this?
2. Write down the cut off frequency/frequencies of the filter.
3. Write the transfer function $H(j\omega)$ of the filter.

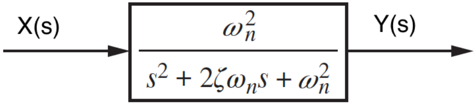
2^{nd} Order Instrument

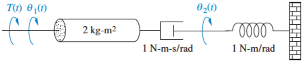
$$a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{d(t)} + a_0y(t) = b_0x(t)$$

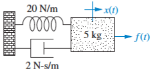
$$H(s) = \frac{Y(s)}{X(s)} = K\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \longrightarrow H(j\omega) = K\frac{\omega_n^2}{-\omega^2+2\xi\omega_n j\omega+\omega_n^2}$$

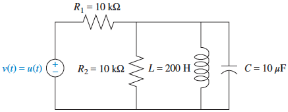
$\frac{a_0}{a_2} = \omega_n$, $\frac{a_1}{2a_0} = \xi$ and $K = \frac{b_0}{a_0}$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}\frac{\omega_n^2}{\sqrt{[1-(\omega/\omega_n)^2]^2+4\xi^2\omega^2/\omega_n^2}}\angle\phi = tan^{-1}\frac{2\xi\omega_n\omega}{\omega_n^2-\omega^2}$$







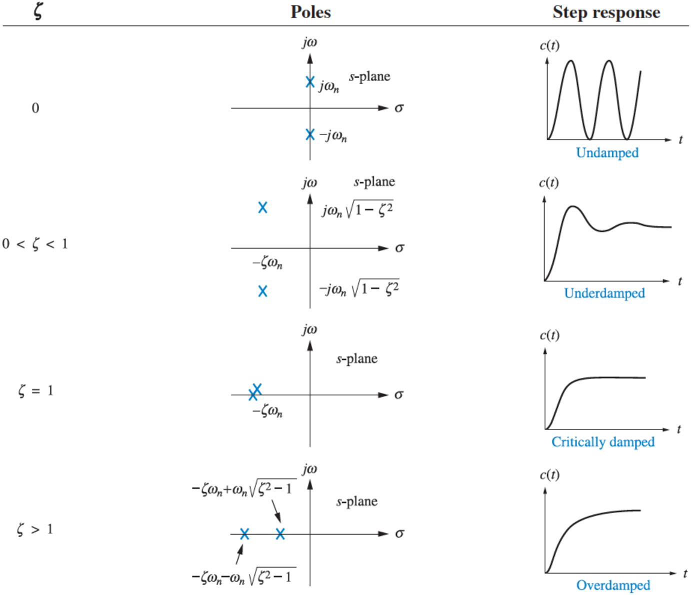


Step Response of a 2^{nd} Order System

1. $\xi > 1 \longrightarrow$ Overdamped
2. $\xi = 1 \longrightarrow$ Critically damped
3. $\xi < 1 \longrightarrow$ Underdamped
4. $\xi = 0 \longrightarrow$ Undamped

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos(\omega_n \sqrt{1-\xi^2}\, t - \phi)$$

where $\phi = tan^{-1}\frac{\xi}{\sqrt{1-\xi^2}}$.


ζ
Poles
Step response
0
jω
s-plane
jω_n
−jω_n
σ
c(t)
t
Undamped
0 < ζ < 1
jω
s-plane
jω_n √(1 − ζ²)
−ζω_n
−jω_n √(1 − ζ²)
σ
c(t)
t
Underdamped
ζ = 1
jω
s-plane
−ζω_n
σ
c(t)
t
Critically damped
ζ > 1
jω
s-plane
−ζω_n+ω_n √(ζ² − 1)
−ζω_n−ω_n √(ζ² − 1)
σ
c(t)
t
Overdamped

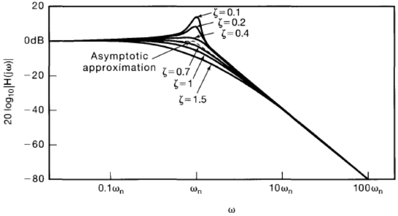
Frequency Response of Second Order Systems

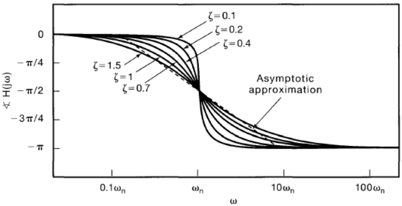
$$H(j\omega) = \frac{\omega_n^2}{-\omega^2+2\omega\xi j+\omega_n^2} = \frac{1}{1-(\omega/\omega_n)^2+2\xi\omega/\omega_n j}$$

$$H(j\omega) = \frac{1}{\{[1-(\omega/\omega_n)^2]^2+(2\xi\omega/\omega_n)^2\}^{1/2}} e^{-\tan^{-1}\frac{2\xi\omega/\omega_n}{1-(\omega/\omega_n)^2}}$$

$$20\log|H(j\omega)| \approx 0 \qquad \text{for } \omega << \omega_n$$

$$20\log|H(j\omega)| \approx -40\log_{10}(\omega/\omega_n) \qquad \text{for } \omega >> \omega_n$$





Modulation

- Often the information signal in instrumentation systems may not be in an optimal form for direct use.
- In such cases, it is used to alter some characteristic of a second signal more suited to the application.
- This process of altering one signal by means of another is called *modulation*; the original information is called the *baseband signal*, and the signal modulated by the baseband signal is termed the *carrier*.
- Reverse process which recovers the original information from the modulated signal is called *demodulation*.

Types of Modulation

1. Amplitude Modulation

$$x(t) = m(t)c(t)$$

$c(t) = A_c \cos(\omega_c t + \phi_c)$: Carrier Signal
$m(t)$: Message Signal

2. Angle Modulation

$$x(t) = A_c \cos(\overbrace{\omega_c t + \phi(t)}^{\theta(t)})$$

$$\phi(t) = \begin{cases} k_p m(t) & \textit{Phase Modulation} \\ k_f \int_{-\infty}^{t} m(\tau)d\tau & \textit{Frequency Modulation} \end{cases}$$

Instantaneous frequency $\omega_i = \frac{d}{dt}\theta(t)$

Hilbert Transform

Analytical Signal

$z(t) = \left(\delta(t) + j\frac{1}{\pi t}\right) * x(t) = x(t) + j\hat{x}(t)$

$\mathcal{F}\{\frac{1}{\pi t}\} = -jsgn(\omega)$

$U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$

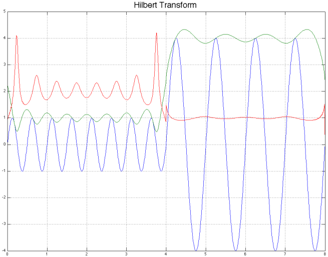
$Z(j\omega) = 2U(\omega)X(j\omega)$

$|z(t)|$: Instantaneous Amplitude of $x(t)$

$\angle z(t) = \tan^{-1}\left(\frac{\hat{x}(t)}{x(t)}\right)$: Instantaneous Phase of $x(t)$

MATLAB SCRIPT for HILBERT TRANSFORM

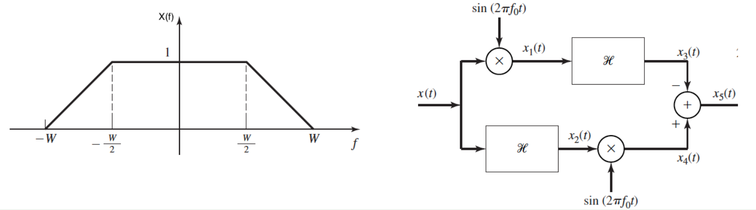
```
%  HILBERT Transform Example
delta_t = 0.01; % sample interval in seconds
A1 = 1; A2 = 4; % amplitudes of sine waves
f1=2; f2= 1; % frequencies of sine waves in Hertz
t1=[0:delta_t:4]'; % time axis for sine wave 1
t2 = [4+delta_t:delta_t:8]'; % time axis for sine wave 2
X = [A1*sin(2*pi*f1*t1 ) ; A2*sin(2*pi*f2*t2)]; % signal generated
Y = hilbert(X); % hilbert transform
plot([t1;t2],[ X abs(Y) 1/(2*pi)*[diff(phase(Y)) ;0]/delta_t ]);grid
xlabel('the original signal (blue) and instantaneous amplitude (red), phase (red) and frequency (green)');
title('Hilbert Transform')
```



the original signal (blue), and instantaneous amplitude (red), phase (red) and frequency (green)

Self Study Question

a) Determine the signal $x_5(t)$ in terms of $x(t)$ and sketch $X_5(f)$.



b) Write a MATLAB script to answer a) by assuming $W = 10$Hz and $f_0 = 10$Hz and $\omega = 2\pi f$.