

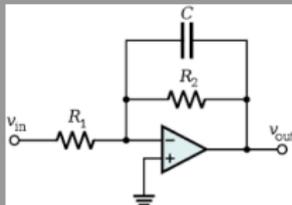
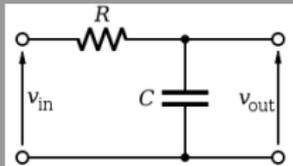
Applications of Laplace and Fourier Transforms

Lecture Notes

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Some concepts and illustrations in this lecture are adapted from the textbook,
N.S. Nise, Control Systems Engineering, 8th edition, Wiley.

FIRST ORDER LOW PASS FILTER



$$V_{out}(s) = V_{in}(s) \frac{1/sC}{R+1/sC}$$

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{sRC+1}$$

$$\frac{V_{in}(s)}{R_1} = -\frac{V_{out}(s)}{R_2 || 1/sC}$$

$$\frac{V_{out}(s)}{V_{in}(s)} = -\frac{R_2}{R_1} \frac{1}{sR_2C+1}$$

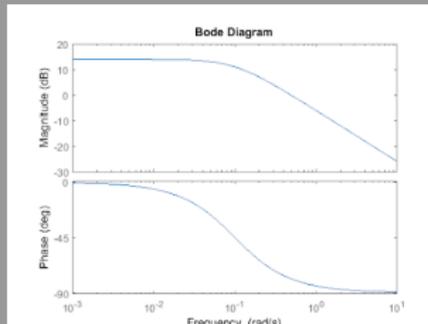
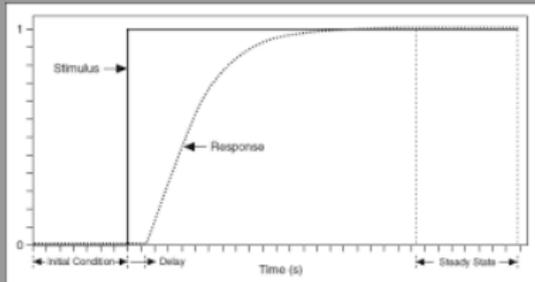
Step Response

$$V_{out}(s) = \frac{1}{s} \frac{1}{sRC+1}, \quad v_{out}(t) = (1 - e^{-t/\tau})u(t),$$

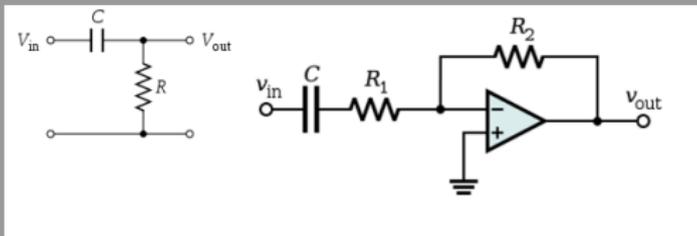
$\tau = RC$: time constant

Frequency Response

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{1}{j\omega\tau+1}$$



FIRST ORDER HIGH PASS FILTER



$$V_{out}(s) = V_{in}(s) \frac{R}{R+1/sC}$$

$$\frac{V_{in}(s)}{R_1+1/sC} = -\frac{V_{out}(s)}{R_2}$$

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{sRC}{sRC+1}$$

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{sR_2C}{sR_2C+1}$$

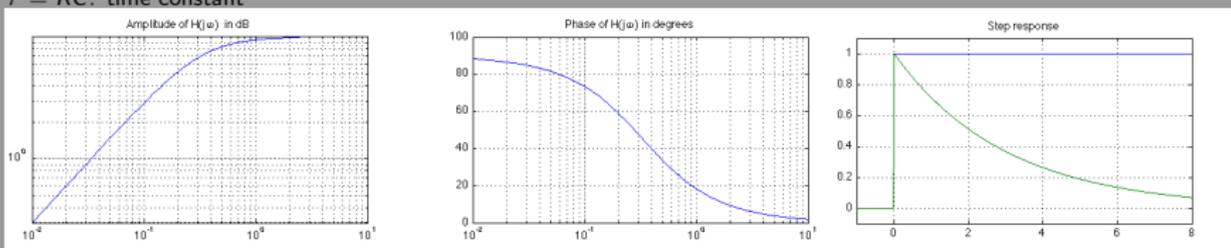
Step Response

$$V_{out}(s) = \frac{1}{s} \frac{sRC}{sRC+1}, \quad v_{out}(t) = e^{-t/\tau} u(t),$$

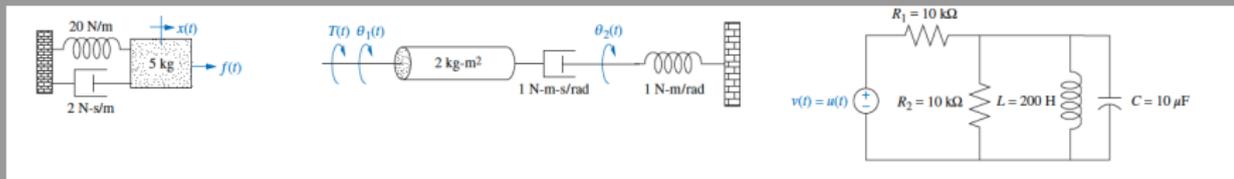
$\tau = RC$: time constant

Frequency Response

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{j\omega\tau}{j\omega\tau+1}$$



SECOND ORDER SYSTEMS $\frac{V_{out}(s)}{V_{in}(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$



$$\frac{d^2}{dt^2}y(t) + a_1\frac{d}{dt}y(t) + a_2y(t) = b_0x(t)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \omega_n : \text{natural frequency}$$

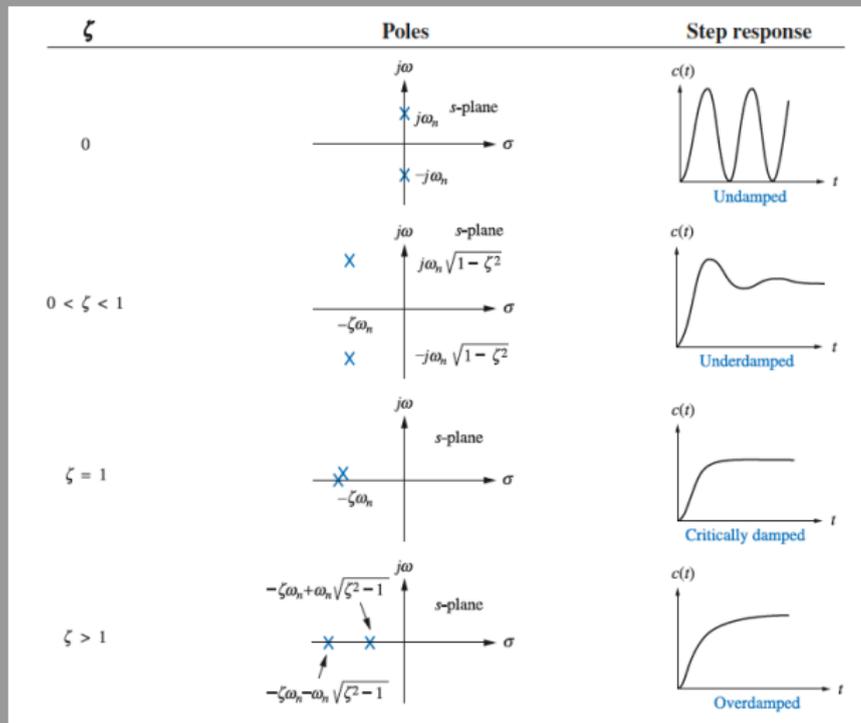
ξ : damping ratio

Step Response of a Second Order System is

$$c(t) = \left(1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1 - \xi^2}} \cos(\omega_n \sqrt{1 - \xi^2} t - \phi)\right) u(t)$$

where $\phi = \tan^{-1} \frac{\xi}{\sqrt{1 - \xi^2}}$.

$$c(t) = \left(1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos(\omega_n\sqrt{1-\xi^2}t - \phi)\right) u(t)$$



Frequency Response of Second Order Systems

$$H(j\omega) = \frac{\omega_n^2}{-\omega^2 + 2\omega\xi j + \omega_n^2} = \frac{1}{1 - (\omega/\omega_n)^2 + 2\xi\omega/\omega_n j}$$

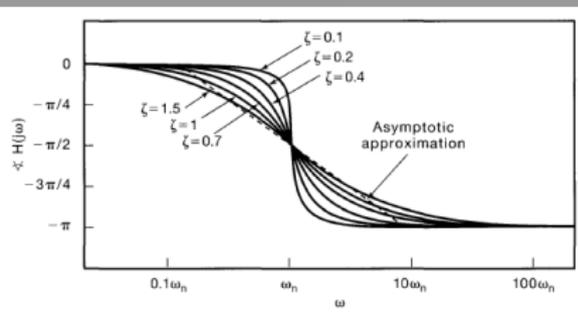
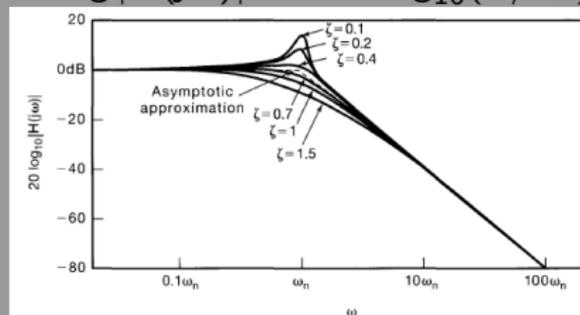
$$H(j\omega) = \frac{1}{\{[1 - (\omega/\omega_n)^2]^2 + (2\xi\omega/\omega_n)^2\}^{1/2}} e^{-j \tan^{-1} \frac{2\xi\omega/\omega_n}{1 - (\omega/\omega_n)^2}}$$

$$20 \log |H(j\omega)| \approx 0$$

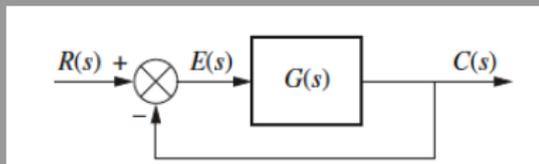
for $\omega \ll \omega_n$

$$20 \log |H(j\omega)| \approx -40 \log_{10}(\omega/\omega_n)$$

for $\omega \gg \omega_n$



Steady-state Error: $e(\infty)$



Let's find the steady-state errors for the test inputs i) $25u(t)$, ii) $37tu(t)$ and iii) $47t^2u(t)$ for a unity feedback system with open

loop transfer function $G(s) = \frac{450(s+8)(s+12)(s+15)}{s(s+38)(s^2+2s+28)}$

i) $e(\infty) = 0$, ii) $e(\infty) = \text{Constant}$ and iii) $e(\infty) = \infty$

Hilbert Transform

Analytical Signal

$$z(t) = \left(\delta(t) + j\frac{1}{\pi t} \right) * x(t) = x(t) + j\hat{x}(t)$$

$$\mathcal{F}\left\{\frac{1}{\pi t}\right\} = -j\text{sgn}(\omega)$$

$$U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$Z(j\omega) = 2U(\omega)X(j\omega)$$

$|z(t)|$: Instantaneous Amplitude of $x(t)$

$\angle z(t) = \tan^{-1} \left(\frac{\hat{x}(t)}{x(t)} \right)$: Instantaneous Phase of $x(t)$

MATLAB SCRIPT for HILBERT TRANSFORM

```
% HILBERT Transform Example
delta_t = 0.01; % sample interval in seconds
A1 = 1; A2 = 4; % amplitudes of sine waves
f1=2; f2= 1; % frequencies of sine waves in Hertz
t1=[0:delta_t:4]'; % time axis for sine wave 1
t2 = [4+delta_t:delta_t:8]'; % time axis for sine wave 2
X = [A1*sin(2*pi*f1*t1 ) ; A2*sin(2*pi*f2*t2)]; % signal generated
Y = hilbert(X); % hilbert transform
plot([t1;t2],[ X abs(Y) 1/(2*pi)*[diff(phase(Y)) ;0]/delta_t ]);grid
xlabel('the original signal (blue) and instantaneous amplitude (red), phase (red) and frequency (green)')
title('Hilbert Transform')
```

